10. *Dimension of the Fixed Point Set of Z_{p^r}-actions*

By Katsuo KAWAKUBO
Osaka University

(Comm. by Kenjiro SHODA, M. J. A., Jan. 12, 1974)

§ 1. **Introduction.** Concerning the dimension of the fixed point set of G-actions, much has been studied [3], [1], [2], [9], [10], [7], and [8]. In this note, we consider a Z_{p^r}-action (M^n, ϕ, Z_{p^r}) on a closed oriented manifold M^n and study the relation between the bordism properties of M^n and the dimension of the fixed point set. If the action is regular, such a problem was studied in [8]. Here we are concerned with general Z_{p^r}-actions.

In order to state the results, we introduce the following notations. Denote by Ω_n the Thom group of all bordism classes $[M^n]$ of closed oriented smooth n-manifold M^n. Let $\Omega(4j)$ be the subring of $\Omega_* \otimes Z_p$ generated by $\{\Omega_0, \Omega_4, \Omega_8, \cdots, \Omega_{4j}\}$. Let $F(Z_{p^r}, k)$ be the subring of $\Omega_* \otimes Z_p$ generated by those bordism classes which are represented by a manifold admitting a Z_{p^r}-action such that the dimension of the fixed point set is less than or equal to k.

Then we have

Theorem. (1) $F(Z_{p^r}, 4k) = F(Z_{p^r}, 4k+1) = \Omega(4kp^r + 2p^r - 2)$

(2) $F(Z_{p^r}, 4k+2) = F(Z_{p^r}, 4k+3) = \Omega(4kp^r + 4p^r - 4)$.

Remark. If $k = -1$, then Theorem means the main result of Conner-Floyd [4].

Corollary 1. *Let (M, Z_{p^r}) be a Z_{p^r}-action. If $[M]$ is indecomposable in $\Omega_* \otimes Z_p$, then there exists a component of the fixed point set of dimension greater than or equal to*

$$\frac{\dim M}{p^r} - 2.$$

Corollary 2. *Each element $x \in \Omega_m$ has a representative which admits a Z_{p^r}-action with fixed point set of dimension less than or equal to m/p^r.*

Throughout this paper, p denotes an odd prime integer.

The results in this paper are oriented bordism versions of the excellent papers [5], [7] of tom Dieck.

Detailed proof will appear elsewhere.

§ 2. **Outline of the proof.** The following diagram is an oriented bordism version of tom Dieck [5],

$$\begin{array}{ccccc} G\Omega_*^{Z_{p^r}} & \xrightarrow{i} & \Omega^*_{Z_{p^r}} & \xrightarrow{\alpha} & \Omega^*(BZ_{p^r}) \\ F\downarrow & & \lambda\downarrow & & \Lambda\downarrow \\ \Omega_*\Big(\prod_j BU(n_j)\Big) & \longrightarrow & S^{-1}\Omega^*_{Z_{p^r}} & \xrightarrow{S^{-1}\alpha} & S^{-1}\Omega^*(BZ_{p^r}) \end{array}$$

where $G\Omega_*^{Z_{p^r}}$ denotes the geometric bordism of oriented Z_{p^r}-manifolds. Let $\Omega^*(BZ_{p^r})\xrightarrow{\pi}\Omega^*$ be the map induced by the map: one point $\to BZ_{p^r}$ and $D:\Omega^*\to\Omega_{-*}$ be the Atiyah-Poincaré duality and $\pi':\Omega_*\to\Omega_*\otimes Z_p$ be the projection map. Then it is easy to see

Lemma 1. *By the composition of the following maps*

$$G\Omega_n^{Z_{p^r}}\xrightarrow{i}\Omega^{-n}_{Z_{p^r}}\xrightarrow{\alpha}\Omega^{-n}(BZ_{p^r})\xrightarrow{\pi}\Omega^{-n}\xrightarrow{D}\Omega_n,$$

$[M^n,\phi,Z_{p^r}]$ goes onto $[M]$.

Let ξ_∞ be the canonical complex line bundle over $\boldsymbol{CP}_\infty$ and let $\pi_i:\boldsymbol{CP}_\infty\times\boldsymbol{CP}_\infty\to\boldsymbol{CP}_\infty$ be the projection onto the i-th factor, $i=1,2$. If we denote the cobordism Euler class $e(\xi_\infty)$ by T, we have

$$\Omega^*(\boldsymbol{CP}_\infty)\cong\Omega^*[[T]]$$

and

$$\Omega^*(\boldsymbol{CP}_\infty\times\boldsymbol{CP}_\infty)\cong\Omega^*[[T_1,T_2]]$$

where $T_i=\pi_i^*(T)$. Hence we get a formal group law $F(T_1,T_2)$ by setting

$$F(T_1,T_2)=e(\xi_\infty\hat{\otimes}\xi_\infty)=\sum_{i,j}c_{ij}T_1^iT_2^j$$

where $c_{ij}\in\Omega^{2-2i-2j}$ [11]. If i is an integer, let $[i]_F(T)$ be the operation of "multiplication by i" for the formal group. Let $j:Z_{p^r}\to S^1$ be the natural inclusion, which induces $Bj:BZ_{p^r}\to BS^1\cong\boldsymbol{CP}_\infty$. By making use of the map $(Bj)^*:\Omega^*(\boldsymbol{CP}_\infty)\to\Omega^*(BZ_{p^r})$, we have

$$\Omega^*(BZ_{p^r})\cong\Omega^*[[T]]/[p^r]_F(T).$$

Moreover it is seen by the method of [6] that the Ker $\Lambda\cdot(Bj)^*$ is the ideal

$$[p^r]_F(T)/[p^{r-1}]_F(T).$$

Since $[p^r]_F(T)/[p^{r-1}]_F(T)=p+T\cdot G$, where G is a power series in T, we have

Lemma 2. $D\pi\Lambda^{-1}(0)=p\Omega_n$.

The following lemma will show some of the differences between [7] and our case.

Lemma 3. $S^{-1}\Omega^*_{Z_{p^r}}\cong\Omega_*\Big(\prod_j BU\Big)\otimes Z[V_j,V_j^{-1}]$. *Here* $1\leqq j\leqq(p^r-1)/2$ *and V_j corresponds to the Euler class of the* 1*-dimensional complex vector space on which* $\exp 2\pi i/p^r$ *acts by multiplication with* $\exp 2\pi ji/p^r$.

By combining Lemma 2 and Lemma 3, we have

Lemma 4. *The composition $\pi'\cdot D\cdot\pi\cdot\Lambda^{-1}\cdot S^{-1}\alpha$ induces a well-defined ring homomorphism,*

$$\beta:\text{Image }\lambda\to\Omega_*\otimes Z_p.$$

Let $A:\Omega_*\left(\prod_j BU\right)\otimes Z[V_j, V_j^{-1}]\to\Omega_*\left(\prod_j BU\right)\otimes Z\left[\frac{1}{2}\right][V_j, V_j^{-1}]$ be the map induced by the inclusion $Z\to Z\left[\frac{1}{2}\right]$. It follows from Lemma 4 that β induces a map

$$\beta' : \text{Image } \lambda\otimes Z\left[\frac{1}{2}\right]\to\Omega_*\otimes Z_p\otimes Z\left[\frac{1}{2}\right]\cong\Omega_*\otimes Z_p.$$

Since $Z\left[\frac{1}{2}\right]$ is a flat Z-module, the map

$$A\otimes 1 : \text{Image } \lambda\otimes Z\left[\frac{1}{2}\right]\to\Omega_*\left(\prod_j BU\right)\otimes Z\left[\frac{1}{2}\right][V_j, V_j^{-1}]\otimes Z\left[\frac{1}{2}\right]$$
$$\cong\Omega_*\left(\prod_j BU\right)\otimes Z\left[\frac{1}{2}\right][V_j, V_j^{-1}]$$

is injective and $(A\otimes 1)\left(\text{Image } \lambda\otimes Z\left[\frac{1}{2}\right]\right)=\text{Image }(A\lambda)$. Therefore we have shown the following

Lemma 5. *There exists a ring homomorphism*

$$\beta'' : \text{Image }(A\lambda)\to\Omega_*\otimes Z_p$$

such that $\beta''\cdot A\cdot\lambda=\pi'\cdot D\cdot\pi\cdot\alpha$.

Let F_k be the subring of $\Omega_*\left(\prod_j BU\right)\otimes Z\left[\frac{1}{2}\right][V_j, V_j^{-1}]$ generated by $\bigoplus_{i\leq k}\Omega_i\left(\prod_j BU\right)\otimes Z\left[\frac{1}{2}\right][V_j^{-1}]$.

Put $D_k=F_k\cap\text{Image }(A\cdot\lambda)$.

We now prove the formula (1) of Theorem.

By choosing simply the tom Dieck's examples of dimension zero mod 4 [7], we have

Lemma 6. *There are Z_{p^r}-actions (M_j, Z_{p^r}) $j=1,2,\cdots, kp^r+(p^r-1)/2$, such that*

(1) $\dim M_j=4j$

(2) $[M_j]$ *is a generator of the polynomial ring* $\Omega_*\otimes Z_p$,

(3) $A\lambda i[M_j, Z_{p^r}]\in D_{4k}$.

It follows from Lemma 6 that

$$F(Z_{p^r}, 4k)\supset\Omega(4kp^r+2p^r-2).$$

Suppose that there exists a Z_{p^r}-action (M, Z_{p^r}) such that $[M]$ is in $F(Z_{p^r}, 4k)$ but not in $\Omega(4kp^r+2p^r-2)$. Since $\Omega_*\otimes Z_p$ is the polynomial ring over Z_p,

$$[M], [M_1], [M_2], \cdots, [M_{kp^r+(p^r-1)/2}],$$

are algebraically independent over Z_p. On the other hand, D_{4k} has transcendence degree at most $kp^r+(p^r-1)/2$. By combining Lemma 5 and Lemma 6, we have already found $kp^r+(p^r-1)/2$ independent elements in D_{4k}. Therefore

$$A\lambda i[M, Z_{p^r}], A\lambda i[M_1, Z_{p^r}], \cdots, A\lambda i[M_{kp^r+(p^r-1)/2}, Z_{p^r}],$$

are algebraically dependent. In view of Lemma 5, this means $[M]$, $[M_1], \cdots, [M_{kp^r+(p^r-1)/2}]$ are algebraically dependent, contradicting the assumption.

The proof of the formula (2) of Theorem will be shown quite similarly.

Corollary 1 and Corollary 2 will follow from Theorem directly.

Remark. Professor Tammo tom Dieck kindly informed me that results in this paper can be generalized to the case of arbitrary abelian p-group actions.

References

[1] Boardman, J. M.: On manifolds with involution. Bull. A. M. S., **73**, 136–138 (1967).

[2] Conner, P. E.: Seminar on Periodic Maps. Lecture Notes in Math., **46**. Springer-Verlag (1967).

[3] Conner, P. E., and Floyd, E. E.: Differentiable Periodic Maps. Springer-Verlag (1964).

[4] ——: Maps of odd period. Ann of Math., **84**, 132–156 (1966).

[5] tom Dieck, T.: Bordism of G-manifolds and integrality theorems. Topology, **9**, 345–358 (1970).

[6] ——: Kobordismentheorie klassifizierender Räume und Transformationsgruppen. Math. Z., **126**, 31–39 (1972).

[7] ——: Periodische Abbildungen unitärer Mannigfaltigkeiten. Math. Z., **126**, 275–295 (1972).

[8] Kawakubo, K.: The Index and the Generalized Todd Genus of Z_p-actions (to appear in Proc. Intern. Conference on Manifolds and Related Topics in Topology). Tokyo (1973).

[9] Kawakubo, K., and Uchida, F.: On the index of a semi-free S^1-action. J. Math. Soc. Japan, **23**, 351–355 (1971).

[10] Ossa, E.: Cobordismustheorie von Fixpunktfreien und Semifreien S^1-Mannigfaltigkeiten (Thesis). Bonn (1969).

[11] Quillen, D.: Elementary proofs of some results of cobordism theory using Steenrod operations. Advance in Math., **7**, 29–56 (1971).