5. The Asymptotic Eigenvalue Distribution for Non-smooth Elliptic Operators

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1. Introduction.

The purpose of this note is to study the asymptotic eigenvalue distribution for the following equation

(1.1) $$Au+ru=\lambda pu \qquad r\geq 0.$$

Here A is a positive elliptic differential operator with constant coefficients defined on R^n and $p(x)$ is a positive function. When A is a homogeneous elliptic operator with a non-smooth $p(x)$, the distribution of the eigenvalues of (1.1) was discussed in Birman and Solomjak [3], Birman and Borzov [4] and Rosenbljum [5]. In this note we will study the asymptotic distribution including the case that A is an inhomogeneous operator. The obtained results can be applied to the operator with a large parameter $h>0$

(1.2) $$Au-hp(x)u=\mu u.$$

In fact, it was shown in Birman [2] that the number of negative eigenvalues less than r of equation (1.2) coincides with the number of eigenvalues less than h of equation (1.1).

Only the theorems and an outline of proofs are presented here and details will be published elsewhere.

2. Main Theorems.

Let $A(D)=\sum_{|\alpha|\leq m} a_\alpha D^\alpha$ be an elliptic operator with constant coefficients defined on R^n. We suppose that:

(i) $A(\xi)\geq 0 \qquad$ for $\xi\in R^n$;

(ii) $\xi=0$ is the only zero of $A(\xi)$ of even order $m_0\leq m$.

The principal part of $A(D)$ is denoted by $A_0(D)$.

We denote by $K(l,a)$ ($l>0, a>0$) the set of functions $p(x)$ which satisfy the following conditions:

(i) $p(x)$ is decomposed into $p(x)=p_1(x)+p_2(x)$;

(ii) $p_1(x)$ is a positive smooth function with $\lim_{|x|\to\infty} |x|^l p_1(x)=a$;

(iii) $p_2(x)$ is a nonnegative function with compact support;

(iv) $p_2(x)\in L_p$, where $p=1$ if $m\geq n$ and $p>\dfrac{n}{m}$ if $m<n$.

Let $N_r(\lambda)$ be the number of eigenvalues less than λ of equation (1.1).

Theorem 1. *Let $A(D)$ be an elliptic operator satisfying the above assumption and suppose that $r>0$ and that $p(x)$ belongs to $K(l,a)$. Then,*

(i) *if $l>m$,*

$$N_r(\lambda)=(2\pi)^{-n}\omega_0\int_{R^n} p(x)^{n/m}dx\cdot\lambda^{n/m}+o(\lambda^{n/m}) \qquad \omega_0=\text{meas}\,[\xi\,|\,A_0(\xi)\le 1]$$

(ii) *if $l=m$,*

$$N_r(\lambda)=2\pi)^{-n}\omega_0\frac{S}{m}a^{n/m}\lambda^{n/m}\log\lambda+o(\lambda^{n/m}\log\lambda)$$

where S is the surface measure of the $n-1$ dimensional unit sphere if $n\ge 2$ and $S=2$ if $n=1$.

(iii) *if $l<m$,*

$$N_r(\lambda)=(2\pi)^{-n}\frac{S}{n}\int_{R^n}\frac{d\xi}{(A(\xi)+r)^{n/l}}a^{n/l}\lambda^{n/l}+o(\lambda^{n/l})).$$

Theorem 2 [*homogeneous case*]. *Let $A(D)$ be a homogeneous elliptic operator of order m defined on R^n and suppose that $m<n$ and that $p(x)$ belongs to $K(l,a)$. Then, if $l>m$,*

$$N_0(\lambda)=(2\pi)^{-n}\omega_0\int_{R^n} p(x)^{n/m}dx\lambda^{n/m}+o(\lambda^{n/m}).$$

Remark. Theorem 2 was announced by Rosenbljum [5] without detailed proofs.

Theorem 3 [*inhomogeneous case*]. *Let $A(D)$ be an inhomogeneous elliptic operator satisfying the above assumption and suppose that $p(x)$ belongs to $K(l,a)$.*

(i) *The case $m<n$:*

if $m_0<l<m$,

$$N_0(\lambda)=(2\pi)^{-n}\frac{S}{n}\int_{R^n} A(\xi)^{-n/l}d\xi a^{n/l}\lambda^{n/l}+o(\lambda^{n/l}),$$

if $l>m$,

$$N_0(\lambda)=(2\pi)^{-n}\omega_0\int_{R^n} p(x)^{n/m}dx\lambda^{n/m}+o(\lambda^{n/m}).$$

(ii) *The case $m\ge n$*

if $m_0<l<n\le m$,

$$N_0(\lambda)=(2\pi)^{-n}\frac{S}{n}\int_{R^n} A(\xi)^{-n/l}d\xi a^{n/l}\lambda^{n/l}+o(\lambda^{n/l}).$$

Remark. Under proper conditions, Theorems 1, 2 and 3 can be extended to elliptic operators with variable coefficients.

3. Outline of the proofs.

Sketch of the proof of Theorem 1.

Here we consider only the case that $m>n$ and $l>n$. The general case can be reduced to this case. For the sake of simplicity, we assume that $p(x)$ is a positive smooth function and that $p(x)$ belongs to

$K(l, a)$. Eigenvalue problem (1.1) is transformed to the equivalent eigenvalue problem of the following form

$$p^{-1/2}(A+r)p^{-1/2}v=\lambda v. \tag{3.1}$$

From the assumption that $m>n$ and $l>n$, the operator $p^{1/2}(A+r)^{-1}p^{1/2}$ is a compact operator belonging to trace class. We get the trace formula

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j+\lambda} = \int_{R^n} p(x)A_\lambda(x, x)dx. \tag{3.2}$$

Here $\{\mu_j>0\}_{j=1}^{\infty}$ are eigenvalues of equation (3.1) and $A_\lambda(x, y)$ is an integral kernel of the operator $(A+r+\lambda p)^{-1}$. Following the method developed in Agmon [1], we can estimate $A_\lambda(x, y)$ locally. We get

Lemma 1.

$$\left|A_\lambda(x, x)-(2\pi)^{-n}\int_{R^n}(A(\xi)+r+\lambda p(x))^{-1}d\xi\right|$$
$$\leq \varepsilon\cdot(1+\lambda p(x))^{(n/m)-1}+C(\varepsilon)p(x)^{1/l}(1+\lambda p(x))^{((n-1)/m)-1} \tag{3.3}$$

where ε is any small positive number and $C(\varepsilon)$ is a constant independent of λ and x.

Combining the above Lemma and the Tauberian theorem of Hardy and Littlewood, we get Theorem 1.

Sketch of the proof of Theorem 2.

Here we suppose that $n>m>\frac{n}{2}, l>m$ and that $p(x)$ is a positive smooth function. We begin with the following integral equation (cf. Titchmarsh [6])

$$\frac{1}{\mu_j+\lambda}\varphi_j(x)=p^{1/2}(x)\int_{R^n}K_\lambda(x, y)p^{1/2}(y)\varphi_j(y)dy$$
$$+\frac{\lambda}{\mu_j+\lambda}p^{1/2}(x)\int_{R^n}K_\lambda(x, y)(p(x)-p(y))p^{-1/2}(y)\varphi_j(y)dy$$
$$\equiv a_j(x)+b_j(x) \qquad (j=1, 2, \cdots)$$

where $\{\varphi_i(x)\}_{j=1}^{\infty}$ are eigenfunctions corresponding to eigenvalues $\{\mu_j\}_{j=1}^{\infty}$ and $K_\lambda(x, y)=(2\pi)^{-n}\int_{R^n}\frac{e^{i(x-y)\cdot\xi}}{A(\xi)+r+\lambda p(x)}d\xi$. Estimating $\int_{R^n}\sum_j a_j^2(x)dx$ and $\int_{R^n}\sum_j b_j^2(x)dx$, we get

$$\sum_{j=1} \frac{1}{(\mu_j+\lambda)^2}=C\lambda^{n/m-2}+o(\lambda^{n/m-2}). \tag{3.4}$$

Here the remainder estimate is uniform with respect to r. Combining (3.4) and Tauberian theorem, we obtain Theorem 2.

A similar argument can be applied to the proof of Theorem 3.

References

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