34. *Note on Products of Symmetric Spaces*

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1. Introduction: In [7, Corollary 4.4], we have shown that *if X is a locally compact, symmetric space and Y is a symmetric space, then $X\times Y$ is a symmetric space.*

In this note, we shall show this result is the best possible. Namely, we have

Theorem. *Let X be a regular space. Then the following are equivalent.*

(a): *X is a locally compact, symmetric space.*

(b): *$X\times Y$ is a symmetric space for every symmetric space Y.*

According to A. V. Arhangel'skii [1], a space X is *symmetric*, if there is a real valued, non-negative function d defined on $X\times X$ satisfying the following:

(1): $d(x,y)=0$ whenever $x=y$, (2): $d(x,y)=d(y,x)$, and (3): $A\subset X$ is closed in X whenever $d(x,A)>0$ for any $x\in X-A$.

Metric spaces and semi-metric spaces are symmetric.

We assume all spaces are Hausdorff.

2. Proof of Theorem. For proof, we use the method in [3, Theorem 2.1].

The implication (a)$\Rightarrow$(b) follows from [7, Corollary 4.4].

To prove the implication (b)$\Rightarrow$(a), suppose that $X\times Y$ is a symmetric space for every symmetric space Y, and that a regular space X is not locally compact.

Since a countably compact, symmetric space is compact [5, Corollary 2], X is not a locally countably compact space.

Then there are a point $x_0\in X$ and a local base $\{U_\alpha:\alpha\in A\}$ at x_0 such that each $\overline{U}_\alpha$ is not countably compact. Hence, for each $\alpha\in A$, there is an infinite, discrete closed subset $\{x_i^\alpha: i=1,2,\cdots\}$ of X such that $x_i^\alpha\in\overline{U}_\alpha$.

Topologize A with discrete topology. Let $A_i=A\times\{i\}$ for each positive integer i, and let $\sum_{i=1}^\infty A_i$ be the topological sum of A_i. Let $X_1=\sum_{i=1}^\infty A_i \cup\{\infty\}$ and let $\{V_j(\infty): j=1,2,\cdots\}$ be a local base at the point ∞, where $V_j(\infty)=\{\infty\}\cup\bigcup_{k\geqq j}A_k$. Then a regular space X_1 has a σ-locally-finite base. By J. Nagata and Yu. M. Smirnov Metrization Theorem, X_1 is a metrizable space.

Let $[0,\omega]$ be the ordinal space, where ω is the first countable ordinal number.

Let X_2 be the topological sum $\sum_{\substack{\alpha\in A\\ i\in N}}[0,\omega]\times\{\alpha\}\times\{i\}$, where N is the set of positive integers.

Then X_2 is clearly a metric space.

Let Y be a quotient space obtained by identifying each (α, i) with (ω, α, i), and let f be a quotient map from the topological sum X_1+X_2 onto Y. Then each fiber of f consists of at most two points. Thus a quotient map f on a metric space X_1+X_2 is a Π-map in the sense of [1]. Since the image of a metric space under a quotient, Π-map is symmetrizable [1, Proposition 2.2], Y is a symmetrizable space. (Moreover, we shall remark that Y is a paracompact space.)

By assumption $X\times Y$ is a symmetric space, and hence a k-space, for symmetric spaces are k-spaces [1].
Thus, by [4, Theorem 1.5], $h=i_X\times f$ is a quotient map from $X\times(X_1+X_2)$ onto $X\times Y$.

For each $\alpha\in A$, let

$$S_\alpha=\bigcup_{j\in N}\{(x_j^\alpha, j)\} \quad \text{and} \quad S_{\alpha(i)}=S_\alpha\times\{\alpha\}\times\{i\}.$$

Let $S=\bigcup_{\substack{\alpha\in A\\ i\in N}} h(S_{\alpha(i)})$. Then we see that $h^{-1}(S)$ is closed in $X\times(X_1+X_2)$. Thus S is closed in $X\times Y$, for h is a quotient map.

On the other hand, $(x_0, \infty)\in Cl_{X\times Y}S-S$. Indeed, for any basic open subset

$$O=V(x_0)\times W(\infty) \text{ containing } (x_0, \infty),$$

there is $\alpha_0\in A$ and $j_0\in N$ such that $\overline{U}_{\alpha_0}\subset V(x_0)$ and $(a_0, j_0)\in W(\infty)$.
Since, for each positive integer i, $x_i^{\alpha_0}\in\overline{U}_{\alpha_0}$, there is a positive integr i_0 such that

$$(x_{i_0}^{\alpha_0}, (i_0, \alpha_0, j_0))\in O\cap S.$$

Thus $(x_0, \infty)\in Cl_{X\times Y}S-S$.
Hence S is not a closed subset of $X\times Y$. This is a contradiction. Thus $X\times Y$ is not symmetric space. Hence X is locally compact space. That completes the implication (b)$\Rightarrow$(a).

As a generalization of the first axiom of countability, A. V. Arhangel'skii [1] has introduced the notion of the *gf-axiom of countability*.

That is, a space X satisfies the gf-axiom of countability, if for each $x\in X$, there is a sequence $\{g_i(x)\}_{i=1}^\infty$ of subsets containing x with the following:

A subset O of X is open whenever for each $x\in O$, there is a positive integer i such that $g_i(x)\subset O$.

Symmetric spaces satisfy the gf-axiom of countability, and spaces satisfying the gf-axiom of countability are sequential spaces in the sense of [2].

Using the same argument as in Section 4 of [7], by [6, Theorem

2.2] and [6, Corollary 2.4], we have

Lemma. *Let X and Y satisfy the gf-axiom of countability, and X a regular, locally countably compact space. Then $X \times Y$ satisfies the gf-axiom of countability.*

From Lemma and the implication (b)$\Rightarrow$(a), we have an analogous result for spaces satisfying the gf-axiom of countability. Namely, we have

Theorem. *Let X be a regular space. Then the following are equivalent.*

(a): *X is a locally countably compact spaces satisfying the gf-axiom of countability.*

(b): *$X \times Y$ satisfies the gf-axiom of countability for every space Y satisfying the gf-axiom of countability.*

References

[1] A. V. Arhangel'skii: Mappings and spaces. Russian Math. Surveys, **21**, 115–162 (1966).

[2] S. P. Franklin: Spaces in which sequences suffice. Fund. Math., **57**, 107–115 (1965).

[3] E. Michael: Local compactness and cartesian products of quotient maps and k-spaces. Ann. Inst. Fourier. Grenoble, **18**, 281–286 (1968).

[4] ——: Bi-quotient maps and cartesian products of quotient maps. Ann. Inst. Fourier. Grenoble, **18**, 287–302 (1968).

[5] S. Nedev: Symmetrizable spaces and final compactness. Soviet Math. Dokl., **8**, 890–892 (1967).

[6] Y. Tanaka: On quasi-k-spaces. Proc. Japan Acad., **46**, 1074–1079 (1970).

[7] ——: On symmetric spaces. Proc. Japan Acad., **49**, 106–111 (1973).