31. *Characterization of the Well-Posed Mixed Problem for Wave Equation in a Quarter Space*

By Mikio TSUJI
Department of Mathematics, Kyoto Industrial University

(Comm. by Kinjirô KUNUGI, M. J. A., Feb. 12, 1974)

§1. Introduction. R. Sakamoto [6] and H. O. Kreiss [3] had proved the existence and the uniqueness of a solution for hyperbolic mixed problem in Sobolev space under the uniform Lopatinski's condition. Recently, S. Miyatake [5] obtained the necessary and sufficient condition under which the mixed problem for second order hyperbolic equation with real variable coefficients is L^2-well posed. In the case where the coefficients are constant, R. Sakamoto [7] obtained the necessary and sufficient condition under which we can solve the mixed problem for general higher order hyperbolic equation in C^∞-space.

In this note we try to solve the following hyperbolic mixed problem in $C^\infty(V(t_0))$-space, $V(t_0)=\{(t,x,y);\, t>t_0, x>0, y\in R^{n-1}\}$,

$$(1.1)\quad \begin{cases} \left(D_t^2-D_x^2-\sum_{i=1}^{n-1} D_{y_i}^2\right)u\equiv \Box u=f(t,x,y) & \text{in } V(t_0)\\ (u, D_t u)=(\varphi_0,\varphi_1)=\vec{\varphi}(x,y) & \text{on } V_0(t_0)=\overline{V(t_0)}\cap\{t=t_0\}\\ B(t,y;D_t,D_x,D_y)u=g(t,y) & \text{on } V_1(t_0)=\overline{V(t_0)}\cap\{x=0\}, \end{cases}$$

where $B=D_x+b_0(t,y)D_t+\sum_{i=1}^{n-1} b_i(t,y)D_{y_i}+c(t,y)$, and $D_t=-i\dfrac{\partial}{\partial t}$,

$$D_x=-i\frac{\partial}{\partial x},\ D_y=(D_{y_1},\cdots,D_{y_{n-1}})=-i\left(\frac{\partial}{\partial y_1},\cdots,\frac{\partial}{\partial y_{n-1}}\right).$$

We assume that b_0, b_i $(i=1,\cdots,n-1)$ and c belong to $\mathcal{B}^\infty(R^n)$, and that b_0 and b_i $(i=1,\cdots,n-1)$ are real-valued.

If a solution $u(t,x,y)$ of (1.1) belongs to $C^m(\overline{V(t_0)})$, then

$$(1.2)\qquad D_t^k(Bu)\Big|_{\substack{t=t_0\\ x=0}}=D_t^k g|_{t=t_0},\qquad k=0,1,\cdots,m.$$

If we rewrite (1.2) by using $f,\vec{\varphi}$ and g, we get the compatibility conditions of order m for $f,\vec{\varphi}$ and g.

Definition 1. The mixed problem (1.1) is said to be $\mathcal{E}$-well posed (at $t=t_0$) if the following two properties hold

(E.1) for any $(f,\vec{\varphi},g)\in C^\infty(V(t_0))\times C^\infty(V_0(t_0))^2\times C^\infty(V_1(t_0))$ which satisfy the compatibility conditions of order 2 there exists a unique solution $u(t,x,y)$ of (1.1) in $C^2(V(t_0))$,

(E.2) there exists a positive constant λ such that the value of the solution of (1.1) at $(t_1,x_1,y_1)\in V(t_0)$ depends only on the data in $C_{(t_1,x_1,y_1)}=\{(t,x,y)\in V(t_0);\, t-t_1<-\lambda\,|(x,y)-(x_1,y_1)|\}$.

Definition 2. The mixed problem (1.1) is uniformly $\mathcal{E}$-well posed if for any $t=t_0$ (1.1) is $\mathcal{E}$-well posed.

Theorem 1. *Suppose that the mixed problem* (1.1) *is uniformly $\mathcal{E}$-well posed, and that* $(b_0(t,y), b_1(t,y), \cdots, b_{n-1}(t,y)) \neq (1, 0, \cdots, 0)$ *for any* $(t,y) \in R^n$, *then it holds*

$$(1.3) \qquad {}^{+}\sqrt{\tau^2-\eta^2}+b_0(t,y)\tau+\sum_{i=1}^{n-1} b_i(t,y)\eta_i \neq 0$$

for $\tau=\sigma-i\gamma$ $(\gamma>0)$, $(\sigma, \eta) \in R^n$ *and* $(t,y) \in R^n$,

where ${}^{+}\sqrt{\tau^2-\eta^2}$ *is a root of* $\tau^2-\eta^2$ *with positive imaginary part.*

We can prove this theorem by applying the method of asymptotic expansion which P. D. Lax used in [4]. K. Kajitani [2] proved the similar results for the mixed problem to the first order hyperbolic system.

Conversely, does this necessary condition assure the solvability of the mixed problem (1.1)? For this problem we get the following "Theorem 2". First we state the definition of H-well-posedness.

Definition 3. The mixed problem (1.1) is said to be H-well posed (at $t=t_0$) if for any integer $k>0$ there exist *a* integer m and a positive number γ_k such that for any $(f, \vec{\varphi}, g) \in H_\gamma^m(V(t_0)) \times H^m(V_0(t_0))^2 \times H_\gamma^m(V_1(t_0))$[1] $(\gamma \geqq \gamma_k)$ which satisfy the compatibility conditions of order $(m-3)$ there exists a unique solution $u(t,x,y)$ of (1.1) in $H_\gamma^k(V(t_0))$.

Remark. Assume that (1.1) is H-well posed. Moreover, if (1.1) has the property (E.2) defined in Definition 1, this problem is $\mathcal{E}$-well posed.

Theorem 2. *Suppose that* $\sup b_0(t,y) \leqq 1-\varepsilon$ *where* ε *is a positive constant. Then the mixed problem* (1.1) *is uniformly H-well posed.*

Comparison with the established results with respect to (1.1). S. Miyatake [5] obtained that $-b_0 \geqq (\sum_{i=1}^{n-1} b_i^2)^{1/2}$ is the necessary and sufficient condition for L^2-well-posedness. M. Ikawa [1] showed that, in the case $n=2$, if $b_0=0$ and $b_1 \neq 0$, (1.1) is $\mathcal{E}$-well posed. But, as M. Ikawa considered (1.1) in general domain, his discussion is more precise.

In this note we limit ourselves to give the sketch of the proof of Theorem 2. The detailed proof will be given in a forthcoming paper.

§ 2. Proof of Theorem 2. We treat the case $t_0=0$ and write $V_0=V_0(t_0)$ and $V_1=V_1(t_0)$. Assume $(f, \vec{\varphi}, g) \in H_\gamma^m(V) \times H^m(V_0)^2 \times H_\gamma^m(V_1)$ which satisfy the compatibility conditions of order $(m-3)$, and extend $(f, \vec{\varphi})$ to be $(\tilde{f}, \tilde{\vec{\varphi}}) \in H_\gamma^m(R_+^1 \times R^n) \times H^m(R^n)^2$. We take $\gamma \geqq c > 0$ where c is determined by the following discussions. First we consider the Cauchy problem

1) $H^m(\Omega)$ $(\Omega \subset R^n)$ is usual Sobolev space. $H_\gamma^m(V(t_0)) = \{u(t,x,y);\, e^{-\gamma t}u \in H^m(V(t_0))\}$. $H_\gamma^m(V_1(t_0)) = \{u(t,x,y);\, e^{-\gamma t}u \in H^m(V_1(t_0))\}$.

$$(2.1)\qquad \begin{cases} \square u=\tilde{f}(t,x,y) & \text{in } t>0,\ (x,y)\in R^n, \\ (u,D_t u)=(\tilde{\varphi}_0,\tilde{\varphi}_1)=\vec{\varphi}(x,y) & \text{on } t=0,\ (x,y)\in R^n. \end{cases}$$

Then there exists a unique solution $u_0(t,x,y)$ of (2.1) in $H^m_\gamma(R^1_+\times R^n)$ $(\gamma\geqq c)$. Put $v=u-u_0$, then

$$(2.2)\qquad \begin{cases} \square v=0 & \text{in } V \\ (v,D_t v)=(0,0) & \text{on } V_0 \\ Bv=g(t,y)-\square u_0|_{x=0}\equiv g_0(t,y) & \text{on } V_1. \end{cases}$$

We define $\tilde{g}_0(t,y)$ by

$$\tilde{g}_0(t,y)=\begin{cases} g_0(t,y) & (t>0,\ y\in R^{n-1}) \\ 0 & (t\leqq 0,\ y\in R^{n-1}), \end{cases}$$

then the compatibility conditions of order $(m-3)$ mean that $\tilde{g}_0(t,y)\in H^{m-2}_\gamma(R^n)$ for $\gamma\geqq c$.

Assume (1.1) is H-well posed. Then, if we take m sufficiently large, (2.2) has a unique solution $v(t,x,y)$ in $H^k_\gamma(V)$ $(\gamma\geqq\gamma_k)$. Hence $v_0(t,y)=\lim_{x\to 0}v(t,x,y)$ and $v_1(t,y)=\lim_{x\to 0}(D_x v)\ (t,x,y)$ exist in $H^{k-1/2}_\gamma(V_1)$ and $H^{k-3/2}_\gamma(V_1)$ respectively. And we extend the definition domains of $v_i(t,y)$ $(i=0,1)$ by $v_i=0$ for $t<0$ $(i=0,1)$, then $v_i\in H^{k-(i+1)/2}_\gamma(R^n)$ $(i=0,1)$. Moreover v_0 and v_1 satisfy

$$(2.3)\qquad \begin{aligned} v_1(t,y)&=(2\pi)^{-n}\int_{R^n} e^{it\tau+iy\eta}\,{}^+\sqrt{\tau^2-\eta^2}\,\hat{u}_0(\tau,\eta)d\sigma d\eta \\ &\equiv({}^+\sqrt{D_t^2-D_y^2}\,v_0)\ (t,y), \end{aligned}$$

$$(2.4)\qquad ({}^+\sqrt{D_t^2-D_y^2}+b_0(t,y)D_t+\sum_{i=1}^{n-1}b_i(t,y)D_{y_i}+c(t,y))v_0=g_0(t,y),$$

where $\tau=\sigma-i\gamma$ $(\gamma>0)$, $(\sigma,\eta)\in R^n$ and

$$\hat{u}(\tau,\eta)=\int_{R^n} e^{-it\tau-iy\eta}u(t,y)dtdy.$$

Conversely, if $v_0(t,y)$ and $v_1(t,y)$ satisfy (2.3) and $v_0\in H^k_\gamma(R^n)$, then the Cauchy problem

$$(2.5)\qquad \begin{cases} D_x^2 v=(D_t^2-D_y^2)v & \text{in } x>0,\ (t,y)\in R^n \\ (v,D_x v)=(v_0,v_1) & \text{on } x=0,\ (t,y)\in R^n \end{cases}$$

has a unique solution v in $H^k_\gamma(\Omega)$ where $\Omega=\{(t,x,y);\ x>0,\ (t,y)\in R^n\}$ and the following estimate holds

$$(2.6)\qquad |v|^2_{k,\gamma,\Omega}\leqq\frac{\text{const}}{\gamma}\langle v_0\rangle^2_{k,\gamma,R^n},$$

where

$$|v|^2_{k,\gamma,\Omega}=\sum_{i+|\alpha|=k}\int_\Omega |e^{-\gamma t}\gamma^i(D_t,D_x,D_y)^\alpha v|^2 dtdxdy,$$

$$\langle v_0\rangle^2_{k,\gamma,R^n}=\sum_{i+|\alpha|=k}\int_{R^n}|e^{-\gamma t}\gamma^i(D_t,D_y)^\alpha v_0|^2 dtdy.$$

The inequality (2.6) shows that, if $v_0\equiv 0$ for $t<0$, then $v\equiv 0$ for $t<0$. Therefore, the necessary and sufficient condition for (1.1) to be H-well posed is that for any $g_0\in H^m_\gamma(V_1)$ there exists a unique solution $v_0(t,y)$ of (2.4) in $H^k_\gamma(V_1)$. We prove this fact in the next section.

§ 3. Solvability of pseudo-differential equation. We put

$$r(t,y;\tau,\eta)={}^{+}\sqrt{\tau^2-\eta^2}+b_0(t,y)\tau+\sum_{i=1}^{n-1}b_i(t,y)\eta_i+c(t,y)$$

where $\tau=\sigma-i\gamma$ $(\gamma>0)$, $(\sigma,\eta)\in R^n$ and $(t,y)\in R^n$. And we define $R(t,y;D_t,D_y)$ by

$$Ru=(2\pi)^{-n}\int_{R^n}e^{it\tau+iy\eta}r(t,y;\tau,\eta)\hat{u}(\tau,\eta)d\sigma d\eta. \tag{3.1}$$

We define the adjoint operator R^* of R by

$$(Ru,v)=(u,R^*v)\qquad \text{for any } u,v\in C_0^\infty(R^n),$$

then it follows

$$R^*v=(2\pi)^{-n}\int_{R^n}e^{it\bar{\tau}+iy\eta}r^*(t,y;\bar{\tau},\eta)\hat{v}(\bar{\tau},\eta)d\sigma d\eta, \tag{3.2}$$

$$r^*(t,y;\bar{\tau},\eta)={}^{-}\sqrt{\bar{\tau}^2-\eta^2}+b_0(t,y)\bar{\tau}+\sum_{i=1}^{n-1}b_i(t,y)\eta_i+\bar{c}(t,y)+D_tb_0+\sum_{i=1}^{n-1}D_{y_i}b_i$$

where $\bar{\tau}=\sigma+i\gamma$ $(\gamma>0)$ and ${}^{-}\sqrt{\bar{\tau}^2-\eta^2}$ is a root of $\bar{\tau}^2-\eta^2$ with negative imaginary part. Then we get the following

Proposition 3.1. *Suppose that* $\sup b_0(t,y)\leqq 1-\varepsilon$ *where* ε *is a positive constant. Then there exist positive constants* γ_k $(k=0,\pm 1,\pm 2,\cdots)$ *such that*

1) $\langle Ru\rangle_{k,\gamma,R^n}\geqq c_k\gamma\langle u\rangle_{k,\gamma,R^n}$ for any $u\in H_\gamma^k(R^n)$ and $\gamma\geqq\gamma_k$,

2) $\langle R^*v\rangle_{k,-\gamma,R^n}\geqq c_k\gamma\langle v\rangle_{k,-\gamma,R^n}$ for any $v\in H_{-\gamma}^k(R^n)$ and $\gamma\geqq\gamma_k$,

where c_k *are positive constants.*

Proof. We prove 1) in the case $k=0$. For simplicity we write $\langle u,v\rangle_\gamma=\langle u,v\rangle_{0,\gamma,R^n}$, then it follows

$$\begin{aligned}\langle Ru,u\rangle_\gamma-\langle u,Ru\rangle_\gamma&=\big(({}^{+}\sqrt{\tau^2-\eta^2}-\overline{{}^{+}\sqrt{\tau^2-\eta^2}})\hat{u}(\tau,\eta),\hat{u}(\tau,\eta)\big)\\&\quad-2i\gamma\langle u,b_0u\rangle_\gamma+\langle u,D_tb_0u\rangle_\gamma+\sum_{j=1}^{n-1}\langle u,D_{y_j}b_ju\rangle_\gamma+\langle u,(\bar{c}-c)u\rangle_\gamma.\end{aligned} \tag{3.3}$$

Denote imaginary part of ${}^{+}\sqrt{\tau^2-\eta^2}$ by $f(\tau,\eta)$, then

$$f(\tau,\eta)=\frac{1}{\sqrt{2}}[\{(\gamma^2+\eta^2-\sigma^2)^2+4\sigma^2\gamma^2\}^{1/2}+\gamma^2+\eta^2-\sigma^2]^{1/2}\geqq\gamma.$$

Therefore we get

$$\begin{aligned}\mathrm{Im.}\,\langle Ru,u\rangle_\gamma&\geqq(f(\tau,\eta)\hat{u}(\tau,\eta),\hat{u}(\tau,\eta))-\gamma\langle u,b_0u\rangle_\gamma\\&\quad+\frac{1}{2i}\langle u,D_tb_0u\rangle_\gamma+\frac{1}{2i}\sum_{j=1}^{n-1}\langle u,D_{y_j}b_ju\rangle_\gamma+\langle \mathrm{Im.}\,c\cdot u,u\rangle_\gamma\\&\geqq(\varepsilon\gamma-M_0)\langle u\rangle_\gamma^2\end{aligned}$$

where M_0 is a positive constant. If we put $\gamma_0=2M_0/\varepsilon$, then

$$\langle Ru\rangle_\gamma\geqq\frac{\varepsilon}{2}\gamma\langle u\rangle_\gamma\qquad \text{for any } \gamma\geqq\gamma_0.$$

By the results of Proposition 3.1, we can easily obtain the following

Proposition 3.2. *For any* $g_0(t, y) \in H_\gamma^k(R^n)$ $(\gamma \geqq \gamma_k)$ *there exists a unique solution* $v_0(t, y)$ of (2.4) in $H_\gamma^k(R^n)$ *satisfying*

$$\langle g_0 \rangle_{k,\gamma,R^n} \geqq c_k \gamma \langle v_0 \rangle_{k,\gamma,R^n} \qquad \text{for any } \gamma \geqq \gamma_k.$$

References

[1] M. Ikawa: Mixed problem for the wave equation with an oblique derivative boundary condition. Osaka J. Math., **7**, 495–525 (1970).

[2] K. Kajitani: A necessary condition for the well posed hyperbolic mixed problem with variable coefficients (to appear).

[3] H. O. Kreiss: Initial boundary value problems for hyperbolic systems. Comm. Pure Appl. Math., **23**, 277–298 (1970).

[4] P. D. Lax: Asymptotic solutions of oscillatory initial value problems. Duke Math. J., **24**, 627–646 (1957).

[5] S. Miyatake: Mixed problem for hyperbolic equation of second order (to appear).

[6] R. Sakamoto: Mixed problems for hyperbolic equations. I, II. J. Math. Kyoto Univ., **10**, 349–373, 403–417 (1970).

[7] ——: Hyperbolic mixed problems with constant coefficients (to appear).

[8] T. Shirota: On the propagation speed of hyperbolic operator with mixed boundary conditions. J. Fac. Sci. Hokkaido Univ., **22**, 25–31 (1972).