27. *Riemannian Manifolds Admitting Some Geodesic*

By Tetsunori KUROGI
Fukui University

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1. Introduction. Let M be a compact Riemannian manifold and f an isometry of M. Then a geodesic α on M is called f-invariant geodesic if $f\alpha=\alpha$. It is not known much about isometry invariant geodesic. In this paper we see what kind of Riemannian manifold admits an isometry invariant geodesic. Our results are following;

Theorem A (K. Grove). *Let M be a compact connected, simply connected and oriented Riemannian manifold of odd dimension and f an orientation preserving isometry of M. Then there exists an f-invariant geodesic.*

Theorem B. *Let M be a compact connected, simply connected and oriented Riemannian manifold of $2k$-dimension and f an orientation preserving isometry of M. Then there exists an f-invariant geodesic for $k=1$ and also well for $k>1$ if $\lambda_k(f)=$ even where $\lambda_k(f)$ is the trace of an induced homomorphism $f_k: H_k(M, Q)\to H_k(M, Q)$ where Q is the field of rational numbers.*

Corollary. *Let M be a manifold of Theorem B. Then M admits an f-invariant geodesic for any orientation preserving isometry f of M if $H_k(M, Q)=0$.*

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2. Fixed points of isometry. Let M be a compact manifold and f be an isometry of M. Then the induced homomorphism by f of the i-th homology group of M over coefficient Q is denoted by $f_i: H_i(M, Q)\to H_i(M, Q)$ and the trace of f_i by $\lambda_i(f)$.

Lemma 1. *Let M be an n-dimensional orientable Riemannian manifold and f an orientation preserving isometry, then we have $\lambda_i(f)=\lambda_{n-i}(f)$ $(i=1\sim n)$.*

Proof. We have only to use the Poincaré duality. q.e.d.

Lemma 2. *Let M be an odd dimensional orientable Riemannian manifold and f an orientation preserving isometry of M, then f has no isolated fixed points.*

Proof. Let x be a fixed point of f and $f_*: T_x(M)\to T_x(M)$ be an induced homomorphism by f. Then f_* is an element of $SO(n)$ and so f_* has a following representation with respect to a suitable basis;

$$\begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_p & \\ & & & & I_{n-2p} \end{bmatrix}$$

where

$$A_i=\begin{pmatrix}\cos\theta_i, & -\sin\theta_i \\ \sin\theta_i, & \cos\theta_i\end{pmatrix}, \qquad I_{n-2p}=\begin{bmatrix}1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}.$$

If we take a vector $V=(0, 0, \cdots, 0, v_{2p+1}, \cdots, v_n)$ of $T_x(M)$ with $|V|<\varepsilon$ for any positive ε where $|\cdot|$ is a norm in $T_x(M)$, then $\exp(V)$ is a fixed point of f because of $\exp\cdot f_*=f\cdot\exp$. Therefore x is not an isolated fixed point.

For fixed point of isometry the following theorem is very useful;

Theorem C (S. Kobayashi). *Let M be a compact Riemannian manifold and f an isometry of M. Let F be the fixed point set of f. If we denote the Lefschetz number of f by $L(f)$ and the Euler number of F by $\chi(F)$, then $L(f)=\chi(F)$.*

3. Proof of theorems. K. Grove has obtained the following result in his paper [1].

Theorem D (K. Grove). *Let M be a compact connected and simply connected Riemannian manifold. Then for any isometry f of M without or with at least two fixed points M admits an f-invariant geodesic.*

By this theorem and Lemma 2 we have Theorem A. Now we go to the proof of Theorem B. The case of $k>1$: By Lemma 1 we have $L(f)=2\sum_{i=2}^{k-1}(-1)^i\lambda_i(f)+(-1)^k\lambda_k(f)+2$. Thus if $\lambda_k(f)=\text{even}$, $L(f)\neq 1$. Therefore we have $\chi(F)\neq 1$. If $\chi(F)=0$ there are no fixed points or F contains non isolated fixed point set. Hence M admits an f-invariant geodesic by Theorem D. The case of $k=1$: By assumption of Theorem B we have $L(f)=2$, therefore $\chi(F)=2$. So F consists of at least two fixed points. Thus by virtue of Theorem D M admits an f-invariant geodesic. This completes the proof of Theorem B. In particular $\lambda_k(f)=0$ is an optimal solution of the equation $L(f)\neq 1$ and so corollary follows.

References

[1] Grove, K.: Condition (C) for the energy integral on certain path-spaces and applications to the theory of geodesics (preprint series of Aarhus Universitet (1970)).

[2] Kobayashi, S.: Transformation groups in differential geometry. Ergebnisse der Math., Bd, **70** (1972).