# 26. *On Bounded Reinhardt Domains*

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(Comm. by Kunihiko KODAIRA, M. J. A., Feb. 12, 1974)

§ 1. Let $D$ be a connected bounded Reinhardt domain in $n$-complex Euclidean space $\boldsymbol{C}^n$ which contains the origin $o$, and $\mathrm{Aut}(D)$ be the group of all biholomorphic transformations of $D$ onto itself. The identity component of $\mathrm{Aut}(D)$ is denoted as $\mathrm{Aut}^0(D)$ $(=G)$. (For any bounded domain, it is a classical theorem of H. Cartan that $\mathrm{Aut}^0(D)$ becomes a Lie transformation group of $D$ in a natural manner.)

When $n=2$, P. Thullen [5] determined the bounded Reinhardt domains with the property such that $G\cdot o \supsetneqq \{o\}$, ($G\cdot o$ denotes the $G$-orbit of the origin). In fact, such domains are holomorphically equivalent to the polydisc $\{(z,w)\in \boldsymbol{C}^2\,;\,|z|<1, |w|<1\}$, or to the *Thullen domain* $\{(z,w)\in \boldsymbol{C}^2\,;\,|z|^2+|w|^\alpha<1\ (\alpha>0)\}$. Recently, I. Naruki [4] and M. Ise [2] have treated a class of Reinhardt domains containing the higher-dimensional generalization of Thullen domains.

In this note, we intend to generalize these works and, further, to classify bounded Reinhardt domains in the $n$-dimensional case from the group theoretic point of view. The full exposition will be given elsewhere. The author is gratefull to Prof. Mikio Ise for suggesting the present problem and for his advices.

§ 2. Throughout this note, $D$ will represent a bounded Reinhardt domain in $\boldsymbol{C}^n$, $\mathfrak{g}(D)$ the Lie algebra of complete holomorphic vector fields, and $\mathfrak{k}(D)$ the subalgebra of $\mathfrak{g}(D)$ which consists of all elements vanishing at the origin. Then, $\mathfrak{g}(D)$ can be identified canonically with the Lie algebra of $G$ $(=\mathrm{Aut}^0(D))$ where $\mathfrak{k}(D)$ corresponds to that of the isotropy subgroup $K$ of $G$ with respect to the origin.

Since $D$ is a circular domain, $K$ consists of linear transformations of $\boldsymbol{C}^n$ and a transformation defined by $k_\theta : z \to e^{i\theta}z$ $(\theta \in \boldsymbol{R}, z \in \boldsymbol{C}^n)$ belongs to the center of $K$.

Now, we can write the vector field $X$ contained in $\mathfrak{g}(D)$ in the form:

$$X=\sum p^k(\partial/\partial z^k),$$

where $z^1,\cdots,z^n$ denote the coordinates in $\boldsymbol{C}^n$, and $p^k$ $(k=1,\cdots,n)$ holomorphic functions on $D$. A vector field $X$ is said to be a polynomial vector field if the components $p^k$ of $X$ are polynomials of $z^1,\cdots,z^n$ and $X$ is homogeneous of degree $\lambda$, if each $p^k$ is a homogeneous polynomial of degree $\lambda$. For example, a vector field defined by $\partial=\sum\sqrt{-1}z^k(\partial/\partial z^k)$

is homogeneous of degree 1 and belongs to $\mathfrak{g}(D)$, because it generates the one parametric group $\{k_\theta\}_{\theta\in\boldsymbol{R}}$.

In analogy with the Cartan decomposition of a semisimple Lie algebra, we have the following lemma. The proof proceeds along the same line as that of [3].

**Lemma 1.** i) *Let $J$ be an endomorphism of $\mathfrak{g}(D)$ defined by $J(X)=[\partial, X]$. Then $\mathfrak{k}(D)$ coincides with the kernel of $J$.*

ii) *If we put $\mathfrak{p}=\{X\in\mathfrak{g}(D)\,;\,J^2X=-X\}$, then we have $\mathfrak{g}(D)=\mathfrak{k}(D)+\mathfrak{p}$.*

iii) *Let $X\in\mathfrak{g}(D)$ and let $X=\sum_\lambda X_\lambda$ be an expansion by homogeneous polynomial vector fields near the origin (each $X_\lambda$ is homogeneous of degree $\lambda$). Then $X_\lambda=0$ for $\lambda>2$, and $X_0\in\mathfrak{p}^-, X_1\in\mathfrak{k}(D)$ and $X_2\in\mathfrak{p}^+$, where $\mathfrak{p}^\pm=\{X\in\mathfrak{p}\otimes\boldsymbol{C}\,;\,JX=\pm\sqrt{-1}X\}$.*

iv) $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}, [\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}, [\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}, [\mathfrak{p}^\pm,\mathfrak{p}^\pm]=0, [\mathfrak{k}^C,\mathfrak{p}^\pm]\subset\mathfrak{p}^\pm$, where $\mathfrak{k}^C=\mathfrak{k}\otimes\boldsymbol{C}$.

**Remark.** We note, that for any bounded circular domain, the above lemma is valid.

§ 3. The following lemma is derived immediately from elementary properties of Reinhardt domains.

**Lemma 2.** i) *The polynomial vector fields $\sqrt{-1}z^i(\partial/\partial z^i)$ $(i=1,\cdots,n)$ are contained in $\mathfrak{g}(D)$, and $\mathfrak{h}_0=\sum_i\boldsymbol{R}(\sqrt{-1}z^i\partial/\partial z^i)$ is a maximal abelian subalgebra of $\mathfrak{k}(D)$.*

ii) *Moreover, the complexification $\mathfrak{h}=\mathfrak{h}_0\otimes\boldsymbol{C}$ is a Cartan subalgebra in the following sense:* 1) *$\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}^C=(\mathfrak{g}\otimes\boldsymbol{C})$,* 2) *for any $H\in\mathfrak{h}$, ad $H$ is a semisimple endomorphism of $\mathfrak{g}^C$.*

From this lemma, we can consider the root decomposition of $\mathfrak{g}^C$ with respect to $\mathfrak{h}$. As stated in [1], if $\alpha$ is a linear form on $\mathfrak{h}$, we define

$$\mathfrak{g}^\alpha=\{X\in\mathfrak{g}^C\,;\,[H,X]=\alpha(H)X \qquad \text{for all } H\in\mathfrak{h}\},$$

and we call $\alpha$ a *root* if $\mathfrak{g}^\alpha\neq\{0\}$. We denote by $\Delta$ the set of non-zero roots of $\mathfrak{g}^C$.

**Lemma 3.** i) *For any non-zero root $\alpha$, $\dim_C\mathfrak{g}^\alpha=1$.*

ii) $\mathfrak{g}^C=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$.

iii) *If $\mathfrak{g}^\alpha\subset\mathfrak{p}^-$, then $\alpha=-\alpha_k$ for some $k(\leqq n)$, where $\alpha_k$ is defined by $\alpha_k(z^j\partial/\partial z^j)=\delta^j_k$ $(j=1,\cdots,n)$.*

iv) $\mathfrak{g}^{-\alpha_k}=\boldsymbol{C}(\partial/\partial z^k), \mathfrak{g}^{\alpha_k}=\boldsymbol{C}z^k(\sum a^iz^i\partial/\partial z^i)\subset\mathfrak{p}^+$.

v) *If $\mathfrak{g}^\alpha\subset\mathfrak{k}^C$, then $\alpha=\alpha_k-\alpha_h$ for some $k,h$ and $\mathfrak{g}^{\alpha_k-\alpha_h}=\boldsymbol{C}z^k(\partial/\partial z^h)$.*

From iii), iv), permuting the coordinate axis, we can assume $\mathfrak{p}^-=\sum_{i=1}^s\boldsymbol{C}(\partial/\partial z^i)$ $(s\leqq n)$. We denote by $\tau$ the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$, and for $k\leqq s$ we put

$$\tau(\partial/\partial z^k)=-z^k\Big(\sum_{i=1}^{n} a_i^k z^i \partial/\partial z^i\Big).$$

A matrix $A=(a_i^k)$ of $(s,n)$ type will be called the type matrix of $D$. As a corollary of Lemma 3, we have an information of $A$.

**Corollary.** i) *Each element $a_i^k$ is non negative.*

ii) *$a_k^k>0$ for $k\leqq s$.*

iii) *For $k,h\leqq s, \alpha_k-\alpha_h$ is a non-zero root if and only if $(a_k^k)^{-1}a_h^k=(a_h^h)^{-1}a_k^h=1$. In this case $(a_k^k)^{-1}a_i^k=(a_h^h)^{-1}a_i^h$ for $i=1,\cdots,n$. In another case, $a_h^k=a_k^h=0$.*

iv) *If $\alpha_k-\alpha_h$ is a non-zero root for $k,h>s$, then we have $a_k^l=a_h^l$ for $l\leqq s$.*

v) *For $k\leqq s, h>s, \alpha_k-\alpha_h$ can not be a root.*

Therefore, we can assume that $a_k^k=1$ for all $k\leqq s$, if necessary, we transform: $(z^1,\cdots,z^n)\to(a_1^1z^1,\cdots,a_s^sz^s,z^{s+1},\cdots,z^n)$.

**§ 4.** Let $\varDelta(\mathfrak{k})$ be the set of roots such that $\mathfrak{g}^\alpha\subset\mathfrak{k}^C$. Then, we can endow the set $P=\{\alpha_i; i=1,\cdots,n\}$ with an equivalence relation by setting $\alpha_i\sim\alpha_j$ if $\alpha_i-\alpha_j\in\varDelta(\mathfrak{k})$ or $\alpha_i=\alpha_j$. In view of Corollary, we have the following disjoint decomposition with respect to this relation:

$$P=P_1\cup\cdots\cup P_r\cup P_1'\cup\cdots\cup P_t',$$

where $P_i(1\leqq i\leqq r)$ consists of $\alpha_k$ $(k\leqq s)$ and $P_j'$ $(1\leqq j\leqq t)$ consists of $\alpha_h$ $(h>s)$. According to this decomposition, we define the disjoint decomposition of $\varDelta(\mathfrak{k})$ as follows:

$$\varDelta(\mathfrak{k})=\varDelta_1\cup\cdots\cup\varDelta_r\cup\varDelta_1'\cup\cdots\cup\varDelta_t',$$

where $\varDelta_i=\{\alpha_h-\alpha_l; \alpha_h\neq\alpha_l, \alpha_h,\alpha_l\in P_j\}$ and, $\varDelta_j'=\{\alpha_h-\alpha_l; \alpha_h\neq\alpha_l,\alpha_h,\alpha_l\in P_j'\}$. Further, we define the matrix $B(l)=(b_{ij}^l)$ (resp. $B'(l)=(b_{ij}'^l)$) by

$$\tau(z^i\partial/\partial z^j)=-b_{ij}^l z^j\partial/\partial z^i$$

for $\alpha_i-\alpha_j\in\varDelta_l$ (resp. $\varDelta_l'$). Then we have

**Lemma 4.** *For each $i,j$ with $\alpha_i-\alpha_j\in\varDelta_l$, $b_{ij}^l=1$, and moreover $b_{ij}'^l$ is positive for each $i,j$ with $\alpha_i-\alpha_j\in\varDelta_l'$.*

From the similar arguments as above, we can assume that $b_{ij}'^l=1$. After all, combining these arguments we can find the Reinhardt domain $\tilde{D}$ holomorphically equivalent to $D$ which has the following type matrix

$$\tilde{A}=\left(\begin{array}{cccc|ccc} E_1 & & & O & F_1^1 & \cdots & F_t^1 \\ & \ddots & & & \vdots & \ddots & \vdots \\ & & E_2 & & & & \\ & & & \ddots & & & \\ O & & & E_r & F_1^r & \cdots & F_t^r \end{array}\right),$$

where

$$E_i=\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\in \mathrm{Mat}\,(n_i\times n_i),$$

$$F^i_j=\begin{pmatrix} p^i_j & \cdots & p^i_j \\ \vdots & \ddots & \vdots \\ p^i_j & \cdots & p^i_j \end{pmatrix} \in \mathrm{Mat}\,(n_i \times m_i),$$

and $n_i$ (resp. $m_i$) denotes the number of elements of $P_i$ (resp. $P'_i$).

§ 5. We have now gathered the sufficient informations about the Lie algebra structure of $\mathfrak{g}(D)$ to know the explicit form of $D$. We assume that $D=\tilde{D}$ only for the sake of simplicity. Using the results of preceding sections, we introduce the following notations:

$$z_i=(z^{n_1+\cdots+n_{i-1}+1}, \cdots, z^{n_1+\cdots+n_i}) \qquad \text{for } i \leqq r,$$
$$w_j=(z^{s+m_1+\cdots+m_{j-1}+1}, \cdots, z^{s+m_1+\cdots+m_j}) \qquad \text{for } j \leqq t,$$
$$|z_i|^2=|z^{n_1+\cdots+n_{i-1}+1}|^2+\cdots+|z^{n_1+\cdots+n_i}|^2.$$

We shall now state the main theorems in the present paper.

**Theorem 1.** i) *Let $D_0$ denote the orbit of $G=G(D)$ containing the origin $(=G\cdot o)$. Then,*

$$D_0=\{(z_1, \cdots, z_r, w_1, \cdots, w_t)\,;\, |z_1|^2<1, \cdots, |z_r|^2<1, w_1=\cdots=w_t=0\}.$$

ii) $D_1=\{(w_1, \cdots, w_t) \in \boldsymbol{C}^{n-s}\,;\,(0, \cdots, 0, w_1, \cdots, w_t) \in D\}$ *is a bounded Reinhardt domain in $\boldsymbol{C}^{n-s}$.*

iii) *Moreover we can characterize the domain $D$ by $D_0$ and $D_1$ as follows:*

$$D=\Big\{(z_1, \cdots, z_r, w_1, \cdots, w_t) \in \boldsymbol{C}^n\,;\,(z_1, \cdots, z_r) \in D_0,$$
$$\Big(\frac{w_1}{(1-|z_1|^2)^{p^1_1/2}\cdots(1-|z_r|^2)^{p^r_1/2}}, \cdots, \frac{w_t}{(1-|z_1|^2)^{p^1_t/2}\cdots(1-|z_r|^2)^{p^r_t/2}}\Big) \in D_1\Big\},$$

*where we adopt the notation of § 3 and § 4.*

**Theorem 2.** $\mathrm{Aut}^0(D)$ *consists of transformations of the following type:*

$$\begin{cases} z_i \mapsto (A^i z_i+b^i)(c^i z_i+d^i)^{-1}, \\ w_k \mapsto B^k \prod_{i=1}^{r} (c^i z_i+d^i)^{-p^i_k} w_k, \end{cases}$$

*where* $A^i \in \mathrm{Mat}\ (n_i \times n_i)$, $c^i \in \mathrm{Mat}\,(1 \times n_i)$, $b^i \in \mathrm{Mat}\,(n_i \times 1)$, $d^i \in \mathrm{Mat}\,(1 \times 1)$, $B^k \in U(m_k)$ *(Unitary matrix), and they satisfy the following relations.*

$${}^t\bar{A}^i A^i-{}^t\bar{c}^i c^i=I_{n_i}, \quad {}^t\bar{b}^i b^i-|d^i|^2=-1, \quad {}^t\bar{b}^i A^i-\bar{d}^i c^i=0.$$

§ 6. We shall sketch the proof of Theorem 1. Theorem 2 follows at once from Theorem 1, if we notice that the dimension of the transformation group defined in Theorem 2 coincides with that of $\mathfrak{g}(D)$.

Since $D$ is a Reinhardt domain, it is enough to consider the real domain $D_{\boldsymbol{R}}=D \cap \boldsymbol{R}^n$. Let $(x^1, \cdots, x^n)$ be the coordinate in $\boldsymbol{R}^n$, and we put $X_i=\partial/\partial x^i-\sum_j a^i_j x^i x^j \partial/\partial x^j$. Then the system of vector fields $\xi=\{X_i\}^s_{i=1}$ gives an integrable distribution outside $S=\{(x^1, \cdots, x^n)\,;\,\det(\delta^i_j-a^i_j x^i x^j)=0\}(=\{(x^1, \cdots, x^n)\,;\,(1-|x_2|^2)\cdots(1-|x_r|^2)=0\})$. As a consequence of the Frobenius theorem, we can choose the maximal leaf through each point $(x^1, \cdots, x^n) \in \boldsymbol{R}^n - S$. A leaf $L(x^1, \cdots, x^n)$ of $\xi$ containing $(x^1, \cdots, x^n)$ is bounded if and only if $|x_1|^2 < 1, \cdots, |x_r|^2 < 1$, and $L(x^1, \cdots, x^n) \cap (0) \times \boldsymbol{R}^{n-s}$

$$= \left( \frac{x^{s+1}}{(1-|x_1|^2)^{p_1^1/2} \cdots (1-|x_r|^2)^{p_1^r/2}}, \cdots, \frac{x^n}{(1-|x_1|^2)^{p_t^1/2} \cdots (1-|x_r|^2)^{p_t^r/2}} \right).$$

Therefore, the assertions of the theorem are valid.

## References


[1] S. Helgason: Differential Geometry and Symmetric Spaces. Academic Press, New York and London (1962).

[2] M. Ise: On the Thullen domains and Hirzebruch manifolds. I (to appear).

[3] W. Kaup, Y. Matsushima, and T. Ochiai: On the automorphisms and equivalences of generalized Siegel domains. Amer. J. Math., **92**, 475–498 (1970).

[4] I. Naruki: The holomorphic equivalence problem for a class of Reinhardt domains. Publ. Research Inst. Math. Sciences, Kyoto University, **4**, 527–543 (1968).

[5] P. Thullen: Zu den Abbildungen durch analytische Funktionen mehrerer komplexer Veranderlichen. Die Invariantz des Mittelpunktes von Kreiskörper; Math. Annalen, **104**, 244–259 (1931).