23. *Oscillation Theorems for a Damped Nonlinear Differential Equation*

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In this paper we are concerned with the oscillatory behavior of solutions of the nonlinear differential equation

(A) $$x^{(n)}(t)+q(t)\phi(x^{(n-1)}(t))+p(t)f(x(g(t)))=0.$$

Our main purpose is to extend to equation (A) some of the recent results regarding oscillation of solutions of the differential equation with a time lag

(B) $$x^{(n)}(t)+p(t)f(x(g(t)))=0$$

and the differential equation without a time lag

(C) $$x^{(n)}(t)+q(t)\phi(x^{(n-1)}(t))+p(t)f(x(t))=0.$$

We consider only solutions $x(t)$ of (A) which exist on some half-line $[T_x, \infty)$. A solution $x(t)$ of (A) is said to be oscillatory (or to oscillate) if $x(t)$ has a sequence of zeros $\{t_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} t_k=\infty$; otherwise, a solution is said to be nonoscillatory.

Throughout this paper the following assumptions are assumed to hold:

(a) $f \in C(R)\cap C^1(R-\{0\})$, $R=(-\infty, \infty)$, and
$$xf(x)>0, \quad f'(x)\geqq 0 \quad \text{for all } x\in R-\{0\};$$

(b) $\phi \in C(R)$, and there is a constant $M>0$ such that
$$0<y\phi(y)\leqq My^2 \qquad \text{for all } y\in R-\{0\};$$

(c) $g\in C^1(R^+)$, $R^+=(0,\infty)$, $g(t)\leqq t$, $g'(t)\geqq 0$ for all $t\in R^+$, and $\lim_{t\to\infty} g(t)=\infty$;

(d) $p\in C(R^+)$, and $p(t)>0$ for all $t\in R^+$;

(e) $q\in C(R^+)$, and there is a nonnegative function $m\in C(R^+)$ such that $q(t)\leqq m(t)$ for all $t\in R^+$ and $\lim_{t\to\infty} Q(t,T)=\infty$ for any fixed $T\in R^+$, where
$$Q(t,T)=\int_T^t \exp\left(-M\int_T^s m(u)du\right)ds.$$

Lemma. *Suppose that assumptions* (a)—(e) *hold. If $x(t)$ is a nonoscillatory solution of* (A), *then there is a T such that $x(t)x^{(n-1)}(t)>0$ for all $t\in[T,\infty)$.*

Proof. We may assume that $x(t)>0$ on $[t_0,\infty)$, since a parallel argument holds when $x(t)<0$ on $[t_0,\infty)$. Since $\lim_{t\to\infty} g(t)=\infty$, there is $t_1\geqq t_0$ such that $x(g(t))>0$ on $[t_1,\infty)$. Suppose that there is $t^*\in[t_1,\infty)$ at which $x^{(n-1)}(t^*)=0$. From (A) we see that

$$x^{(n)}(t^*)=-p(t^*)f(x(g(t^*)))<0.$$

It follows that $x^{(n-1)}(t)$ cannot have a zero larger than t^*, hence $x^{(n-1)}(t)$ is eventually of constant sign.

Suppose that $x^{(n-1)}(t)<0$ on $[T,\infty)$. Multiplying (A) by $x^{(n-1)}(t)$, integrating over $[T,t]$ and observing that

$$\int_T^t p(s)f(x(g(s)))x^{(n-1)}(s)ds<0,$$

we find

$$\int_T^t x^{(n)}(s)x^{(n-1)}(s)ds+\int_T^t q(s)\phi(x^{(n-1)}(s))x^{(n-1)}(s)ds>0,$$

from which using (b) we have

$$[x^{(n-1)}(t)]^2>[x^{(n-1)}(T)]^2-2M\int_T^t m(s)[x^{(n-1)}(s)]^2ds.$$

Hence, by the Langenhop inequality [7],

$$[x^{(n-1)}(t)]^2\geqq[x^{(n-1)}(T)]^2\exp\left(-2M\int_T^t m(u)du\right),$$

which, in view of the hypothesis that $x^{(n-1)}(t)<0$ on $[T,\infty)$, yields

$$x^{(n-1)}(t)\leqq x^{(n-1)}(T)\exp\left(-M\int_T^t m(u)du\right).$$

Integrating the above inequality over $[T,t]$, we obtain

$$x^{(n-2)}(t)\leqq x^{(n-2)}(T)+x^{(n-1)}(T)Q(t,T).$$

Since $x^{(n-1)}(T)<0$, by (e), we conclude that

$$\lim_{t\to\infty}x^{(n-2)}(t)=-\infty,\quad\text{and hence}\quad\lim_{t\to\infty}x(t)=-\infty.$$

The contradiction completes the proof of the lemma.

Remark 1. This lemma is essentially that given by Kartsatos and Onose [2, Lemma]. Our proof is based on the method used by Baker for second order ordinary differential equations [1, Lemma 1].

Theorem 1. *In addition to* (a)—(e) *assume that* $q(t)\geqq 0$ *for all* $t\in R^+$, *and that for some* $a>0$

$$(1)\qquad \int_a^{\infty}\frac{dx}{f(x)}<\infty,\qquad \int_{-a}^{-\infty}\frac{dx}{f(x)}<\infty.$$

If

$$(2)\qquad \int^{\infty}[g(t)]^{n-1}p(t)dt=\infty,$$

then, for n even, every solution of (A) *is oscillatory and, for n odd, every solution of* (A) *is either oscillatory or tending monotonically to zero as* $t\to\infty$ *together with its first* $n-1$ *derivatives.*

Theorem 2. *In addition to* (a)—(e) *assume that* $q(t)\geqq 0$ *for all* $t\in R^+$, *and that there exist positive numbers* $M,\lambda_0,\alpha<1$ *such that for* $\lambda\geqq\lambda_0$

(3) $f(\lambda x)\geqq M\lambda^{\alpha}f(x)$ *if* $x>0$ *and* $f(\lambda x)\leqq M\lambda^{\alpha}f(x)$ *if* $x<0$.

If

(4) $$\int^{\infty}[g(t)]^{\alpha(n-1)}p(t)dt=\infty,$$

then the conclusion of Theorem 1 *holds.*

Proof of Theorem 1. Assume that equation (A) has a nonoscillatory solution $x(t)$. In view of Lemma, we may suppose without loss of generality that there is a T such that $x(t)>0$ and $x^{(n-1)}(t)>0$ on $[T,\infty)$. Since this implies that $q(t)\phi(x^{(n-1)}(t))\geqq 0$ on $[T,\infty)$, it follows that

(5) $$x^{(n)}(t)+p(t)f(x(g(t)))\leqq 0.$$

Now, arguing exactly as in the proof of Theorem 2 of [4] or Theorem 1 of [6], we can derive from (5) the contradiction

$$\int^{\infty}[g(t)]^{n-1}p(t)dt<\infty.$$

We omit the details.

Proof of Theorem 2. Apply the techniques used in the proof of Theorem 3 of [4] or Theorem 2 of [5] to derive from (5) the contradiction

$$\int^{\infty}[g(t)]^{\alpha(n-1)}p(t)dt<\infty.$$

Remark 2. Theorems 1 and 2 extend some of the recent results [1], [2], [4–6], [8], [9] regarding oscillation of equations (B) and (C).

Theorem 3. *In addition to* (a)—(e) *assume that there is a constant* $K>0$ *such that*

(6) $$q(t)\geqq -\frac{K}{t} \text{ for all sufficiently large } t.$$

If

(7) $$\int^{\infty}t^{n-1}p(t)dt=\infty,$$

then, for n even, every bounded solution of (A) *oscillates and, for n odd, every bounded solution of* (A) *either oscillates or tends monotonically to zero as $t\to\infty$ together with its first $n-2$ derivatives.*

Proof. Suppose that $x(t)$ is a bounded nonoscillatory solution of (A). As before, we may assume that $x(t)>0$ and $x^{(n-1)}(t)>0$ on $[T,\infty)$. From a lemma of Kiguradze [3, Lemma 2] it follows that

(8) $$(-1)^{j+1}x^{(n-j)}(t)\geqq 0 \qquad \text{on } [T,\infty),\ j=1,2,\cdots,n-1.$$

Since $x'(t)$ is of constant sign on $[T,\infty)$, there exists a finite limit $\lim_{t\to\infty}x(t)=x(\infty)$. If n is even, then $x'(t)\geqq 0$ by (8), and hence $x(\infty)>0$, while, if n is odd, $x'(t)\leqq 0$ by (8), so that $x(\infty)>0$ or $x(\infty)=0$.

Suppose that $x(\infty)>0$. Then, there exist $k>0$ and $T_1\geqq T$ such that $f(x(g(t)))\geqq k$ on $[T_1,\infty)$. It follows from (A) and (b) that

(9) $$x^{(n)}(t)-\frac{KM}{t}x^{(n-1)}(t)+kp(t)\leqq 0 \qquad \text{on } [T_1,\infty).$$

Multiplying (9) by t^{n-1} and integrating over $[T_1,t]$, we obtain

$$\int_{T_1}^{t} s^{n-1}x^{(n)}(s)ds-KM\int_{T_1}^{t} s^{n-2}x^{(n-1)}(s)ds+k\int_{T_1}^{t} s^{n-1}p(s)ds\leqq 0, \tag{10}$$

which, after successive integration by parts, yields

$$\begin{aligned} &t^{n-1}x^{(n-1)}(t)-(n-1)t^{n-2}x^{(n-2)}(t)+\cdots+(-1)^{n-2}(n-1)!\,tx'(t)\\ &\quad+(-1)^{n-1}(n-1)!\,x(t)\\ &\quad-KM\{t^{n-2}x^{(n-2)}(t)-(n-2)t^{n-3}x^{(n-3)}(t)+\cdots\\ &\qquad+(-1)^{n-3}(n-2)!\,tx'(t)+(-1)^{n-2}(n-2)!\,x(t)\}\\ &\quad+k\int_{T_1}^{t} s^{n-1}p(s)ds\leqq C, \end{aligned} \tag{11}$$

where C is a constant. Using (8) and the boundedness of $x(t)$, we conclude from (11) that

$$k\int_{T_1}^{\infty} t^{n-1}p(t)dt<\infty,$$

which contradicts (7). Therefore, we must have $x(\infty)=0$. Clearly, this is possible only when n is odd, and in this case the derivatives $x^{(j)}(t)$, $j=1,\cdots,n-2$, also tend monotonically to zero as $t\to\infty$. This completes the proof.

Remark 3. When $g(t)\equiv t$, Theorem 3 improves a result of Kartsatos and Onose [2, Theorem 1].

Example 1. Theorem 3 implies that all bounded solutions of the equation

$$x^{(4)}-\frac{1}{t}x^{(3)}+\frac{6}{t^{3\alpha+1}}|x|^{\alpha}\operatorname{sgn} x=0, \qquad 0<\alpha\leqq 1,$$

are oscillatory. We note that this equation has an unbounded nonoscillatory solution $x(t)=t^3$.

Theorem 4. *Let $n=2$. In addition to* (a)—(e) *assume that conditions* (1), (6) *and* (7) *are satisfied. Then all solutions of* (C) *are oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (C) such that $x(t)>0$ and $x'(t)>0$ on $[T,\infty)$. We multiply (C) by $t/f(x(t))$, integrate over $[T,t]$ and use (6) to obtain

$$\begin{aligned} &\frac{tx'(t)}{f(x(t))}-(KM+1)\int_{T}^{t}\frac{x'(t)}{f(x(t))}dt\\ &\qquad+\int_{T}^{t}\frac{sf'(x(s))[x'(s)]^2}{[f(x(s))]^2}ds+\int_{T}^{t}sp(s)ds\leqq C, \end{aligned} \tag{12}$$

where C is a constant. Taking the limit as $t\to\infty$ and using (1) and (7) we arrive at the contradiction that $x'(t)<0$ for all sufficiently large t. This completes the proof.

Example 2. The conclusion of Theorem 4 is false for the retarded differential equation (A). In fact, the equation

$$x''(t)-\frac{1}{t}x'(t)+\frac{1}{t^2}[x(t^{1/3})]^3=0$$

has a nonoscillatory solution $x(t)=t$, though conditions (1), (6) and (7) are satisfied.

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