22. Uniqueness in the Cauchy Problem for Partial Differential Equations with Multiple Characteristic Roots

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1. Introduction. We are concerned with the uniqueness theorem in the Cauchy problem for the following type of partial differential equations:

$$Pu \equiv \partial_t^m u + \sum_{|\alpha|+j\leqslant m} a_{\alpha,j}(x,t)\partial_x^\alpha \partial_t^j u = 0, \qquad (x \in R^l).$$

Here we assume $a_{\alpha,j}(x,t)$ are sufficiently smooth functions. In the case where the characteristic roots are simple and the coefficients $a_{\alpha,j}(x,t)$ $(|\alpha|+j=m)$ are all real, A. P. Calderón [1] proved the uniqueness theorem in 1958. When (x,t) is two-dimensional, T. Carleman [2] obtained the same result as early as 1938. S. Mizohata [6] proved the uniqueness in the case of elliptic type of order 4 with smooth characteristic roots. Many authors have studied the uniqueness with at most double smooth characteristic roots ([3], [5], etc.). Then a study for elliptic type with triple characteristic roots, was made by K. Watanabe [10], under the assumption that the multiplicity of the characteristic roots is constant.

The purpose of this note is to announce with a short proof a result on the uniqueness theorem for operators with multiple characteristic roots. A forthcoming article will give a detailed proof. Let us consider the following type of operator:

$$P = P_p(x,t;\partial_x,\partial_t)^m + P_{mp-1}(x,t;\partial_x,\partial_t) + R(x,t;\partial_x,\partial_t),$$

where $m \geqslant 2$ and $p \geqslant 1$. Here we assume that, 1) P_p is a homogeneous partial differential operator of order p with real coefficients, continuously differentiable up to order $l+\max\{mp, 6\}$. Moreover its characteristic roots $\{\lambda_j(x,t;\xi)\}_{1\leqslant j\leqslant p}$ of $P_p(x,t;\xi,\lambda)=0$ are distinct for all real $\xi(\neq 0)$, 2) P_{mp-1} is a homogeneous partial differential operator of order $mp-1$ with real coefficients belonging to $C^{l+\max\{mp-1,5\}}$, 3) R is a partial differential operator of order at most $mp-2$, with bounded measurable coefficients.

Let $\{\lambda_j(x,t;\xi)\}_{1\leqslant j\leqslant p}$ be the characteristic roots of P_p. We introduce the following conditions.

(A) $\quad P_{mp-1}(0,0;\xi,\tau)|_{\tau=\lambda_j(0,0;\xi)} \neq 0 \quad$ for all $\xi \in R^l-\{0\} \quad (1\leqslant j\leqslant p)$

(B$_1$) $\quad P_{mp-1}(x,t;\xi,\tau)|_{\tau=\lambda_j(x,t;\xi)} \equiv 0 \quad$ for all $(x,t,\xi) \in U\times(R^l-\{0\})$ $\quad (1\leqslant j\leqslant p)$

U being a neighbourhood of the origin.

(B_2) (B_1) and $\partial_\tau P_{mp-1}(0,0;\xi,\tau)|_{\tau=\lambda_j(0,0;\xi)}\neq 0$ for all $\xi\in R^l-\{0\}$ $(1\leqslant j\leqslant p)$

Then our result is the following

Theorem. *If $m=2$ and all λ_j satisfy the condition* (A) *or* (B_1), *or if $m\geqslant 3$ and all λ_j satisfy the condition* (A) *or* (B_2), *the solution $u(x,t)\in C^{mp}$ of*

$$\begin{cases}Pu=0\\ \partial_t^j u|_{t=0}=0 \qquad (0\leqslant j\leqslant mp-1)\end{cases}$$

vanishes identically in a neighbourhood of the origin.

2. Some comments to the above new type conditions. When we don't assume the above condition (A), (B_1) or (B_2), the following examples show that we should assume another kind of conditions in order to obtain the uniqueness theorem. First, we give an example of elliptic type.

Example 1 (A. Pliś [9]). Let $l\geqslant 1$, $m\geqslant 6$, and $\frac{m+3}{2}<n\leqslant m-1$, $k>\frac{m-1}{2n-m-3}$, Δ be the Laplacian in $R_x^l\times R_t^1$. There is an operator Q of order at most $2m-2$ and $u(x,t)=u(x_1,t)\in C^\infty$ satisfying

$$\begin{cases}[\Delta^m+P_{2m-1}+t^k(\partial_t+i\partial_{x_1})^m(i\partial_{x_1})^n+Q]u=0,\\ u\equiv 0 \qquad (t\leqslant 0),\end{cases}$$

where P_{2m-1} is an arbitrary operator of order $2m-1$ containing only $\partial_{x_2},\cdots,\partial_{x_l}$, and $u(x,t)$ never vanishes in any neighbourhood of the origin.

Note that the term of order $2m-1$ at the origin is nothing but $P_{2m-1}(0,0;\partial_{x_2},\cdots,\partial_{x_l})$. This shows that neither (A) nor (B_2) is satisfied.

Next, we give an example of hyperbolic type.

Example 2 (L. Hörmander [4]). Let $l\geqslant 1$, $r\geqslant 2$. There exist functions $a(x,t)$ and $u(x,t)=u(x_1,t)\in C^\infty$ satisfying $a(0,0)=0$, and

$$\begin{cases}\partial_t^r u+P_{r-1}u+a(x,t)\partial_{x_1}u=0,\\ u\equiv 0 \qquad (t\leqslant 0),\end{cases}$$

where P_{r-1} is an arbitrary operator of order $r-1$ containing only $\partial_{x_2},\cdots,\partial_{x_l}$, and $u(x,t)$ never vanishes in any neighbourhood of the origin.

3. Outline of the proof of the theorem. In the case under the condition (B_1) or (B_2), we can easily obtain the theorem by applying the result under the condition (A). Thus we give the proof of the theorem under the condition (A).

Reduction to a system of first order. We modify $u\equiv 0$ when $t\leqslant 0$, then u remains as a solution of $Pu=0$. When we perform a Holmgren's transformation, all the conditions in the theorem are invariant. Moreover, modifying the coefficients out of the neighbourhood of the origin, we can assume

$$|P_{mp-1}(x,t;\xi,\tau)|_{\tau=\lambda_j(x,t;\xi)}|\geqslant\delta_0|\xi|^{mp-1},$$

where δ_0 is a positive constant.

Let us reduce the equation to a system of first order regarding $(P_p)^m+P_{mp-1}$ as the principal part, in the same way as S. Mizohata-Y. Ohya [8], then we have

$$\tilde{P}U\equiv D_tU-HU-BU=0,$$

where D_t-H is the principal part of the new equation. Then the characteristic roots of $\det(\mu I-H(x,t;\xi))=0$ can be expanded with respect to $|\xi|^{-1/m}$ in the sense of Puiseux by virtue of the condition (A) and they are distinct. More precisely,

Lemma 3.1. *The characteristic roots* $\{\mu_i^{(j)}\}_{\substack{1\leqslant i\leqslant p\\ 1\leqslant j\leqslant m}}$ *are expanded in the following manner,*

$$\mu_i^{(j)}(x,t;\xi)=\lambda_i(x,t;\xi)+\sum_{k=1}^{\infty}\nu_{i,k}^{(j)}(x,t;\xi)|\xi|^{1-k/m},$$

where $(\nu_{i,1}^{(j)})^m=\sqrt{-1}P_{mp-1}(x,t;\xi,\tau)|_{\tau=\lambda_i(x,t;\xi)}/\prod_{k\neq i}(\lambda_i(x,t;\xi)-\lambda_k(x,t;\xi))^m$ *for* $1\leqslant i\leqslant p$, $1\leqslant j\leqslant m$, *and where* $\nu_{i,k}^{(j)}$ *are homogeneous order* 0 *with respect to* ξ *and belong to* $C_{(x,t)}^{l+5}\times C_\xi^\infty$.

Note that the imaginary part of $\nu_{i,1}^{(j)}$ never vanishes.

Now, let us construct the diagonalizator $\mathcal{N}(x,t;\xi)$ of $H(x,t;\xi)$. Let us put $\mathcal{N}(x,t;\xi)=(n_{ij}(x,t;\xi))$.

Lemma 3.2. *We have*

$$n_{ij}=\prod_{k=j-p[j/p]+1}^{p}(\mu_r^{(s)}-\lambda_k)\Big\{\nu_{r,1}^{(s)}\prod_{k\neq r}(\mu_r^{(s)}-\lambda_k)\Big\}^{m-[j/p]-1}\ \text{mod. order }-1,$$

where $r=i-p\left[\dfrac{i-1}{p}\right]$, $s=\left[\dfrac{i-1}{p}\right]+1$.

Because $\mu_i^{(j)}$ is not homogeneous, $\mathcal{N}(x,t;\xi)$ degenerates near the point at infinity. So the operator with the symbol $\mathcal{M}=\mathcal{N}^{-1}$ is not bounded, but by the detailed consideration we can see that the order of $m_{ij}(x,t;D_x)$, the (i,j)-element of $\mathcal{M}$, is at most $1-\left(1/m\left[\dfrac{i-1}{p}\right]+1\right)$.

The above fact gives us $\|\mathcal{N}U\|\geqslant\text{const.}\|(\Lambda+1)^{-1+1/m}U\|$ if we restrict h sufficiently small.

Energy with a weight function. From now on, we assume $u\not\equiv 0$ in any neighbourhood of the origin.

Operating $\mathcal{N}$ to $\tilde{P}U=0$, we have

$$\mathcal{N}\tilde{P}U=D_t\mathcal{N}U-\mathcal{D}\mathcal{N}U-\mathcal{N}_t'U-(\mathcal{N}H-\mathcal{D}\mathcal{N})U-\mathcal{N}BU=0,$$

where $\mathcal{D}$ is a diagonal matrix whose diagonal elements are $\mu_i^{(j)}$. Let us estimate the energy of $\mathcal{N}\tilde{P}U$ with a weight function $\varphi_n(t)=\left(t+\dfrac{1}{n}\right)^{-n}$, namely $E_n=\int_0^h\varphi_n^2(t)\|\mathcal{N}\tilde{P}U(t)\|^2dt$. Concerning the two terms, $\mathcal{N}_t'U$ and

$(\mathcal{N}H-\mathcal{D}\mathcal{N})U$, we have

$$\|\mathcal{N}_t'U\|\leqslant\text{const.}\ (\|\mathcal{N}U\|+\|(\Lambda+1)^{-1}U\|),$$
$$\|(\mathcal{N}H-\mathcal{D}\mathcal{N})U\|\leqslant\text{const.}\ (\|\mathcal{N}U\|+\|(\Lambda+1)^{-1}U\|).$$

Then a slight modification of the Calderón's argument in [1] (see also S. Mizohata [7]), gives the following proposition.

Proposition. *For large n, we have*

$$E_n\geqslant\text{const.}\left\{\frac{1}{n}\sum_{j=0}^{mp-1}\int_0^h\varphi_n^2(t)\,\|\partial_t^j u(t)\|_{mp-j-1}^2dt\right.$$
$$\left.+n\sum_{j=0}^{mp-1}\int_0^h\varphi_n^2(t)\,\|(\Lambda+1)^{-1+1/m}\partial_t^j u(t)\|_{mp-j-1}^2dt\right\}.$$

On the other hand, since $\mathcal{N}\tilde{P}U=0$, we have $E_n=0$. This is inconsistent with the above inequality, so we have the theorem.

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