48. Approximate Solutions for Some Non-linear Volterra Integral Equations

By Shin-ichi NAKAGIRI and Haruo MURAKAMI
Department of Applied Mathematics, Kobe University

(Comm. by Kinjirô KUNUGI, M. J. A., March 12, 1974)

In this short note we give generalized ε-approximate solutions $x(t\,;\xi,\varepsilon)$ of the following non-linear integral equations of Volterra-type

$$\text{(P)} \qquad x(t)=f(t)+\int_0^t g(t,s,x(s))ds.$$

Under very general assumptions on $f(t)$ and $g(t,s,x)$ similar to the Carathèodory-type, R. K. Miller and G. R. Sell [1] proved the local existence theorem by applying the fixed point theorem of Schauder-Tychonoff. We shall prove that their assumptions in [1] assure the existence of generalized ε-approximate solutions $x(t\,;\xi,\varepsilon)$ of (P) and give some interesting properties of $x(t\,;\xi,\varepsilon)$ which will play an essential role in our sequel paper [3]. As an easy application of our results, we can show another existence proof of a solution of (P).

Let $|x|$ denote the Euclidean norm of a vector x of R^n. For each interval I containing O and each subset K of R^n, we define a space $\mathcal{C}(I\,;k)$ by the set of all continuous functions with domain I and range in K with the compact-open topology. Then $\mathcal{C}[0,\alpha]=\mathcal{C}([0,\alpha]\,;R^n)$ is the Banach space of continuous functions on $[0,\alpha]$ with the norm of uniform convergence. We note that the space $\mathcal{C}[0,\alpha)=\mathcal{C}([0,\alpha)\,;R^n)$ is not a Banach space but a Frèchet space. Denote by $\mathcal{L}^1[0,\alpha]$ the Banach space consisting of all Lebesgue measurable functions $x:[0,\alpha]\to R^n$ with finite norm $\int_0^\alpha |x(t)|\,dt<\infty$.

We assume the following hypotheses which are somewhat weaker than those in [1].

(H1) The function f is defined and continuous for all t in $R^+=\{t\in R: t\geqq 0\}$ with values in R^n.

(H2) Let $g(t,s,x)$ be a function defined on $R^+\times R^+\times R^n$ with values in R^n such that

(i) for each fixed $(t,x)\in R^+\times R^n$, $g(t,s,x)$ is Lebesgue measurable in s and $g(t,s,x)=0$ for $s>t$, and

(ii) for each fixed $(t,s)\in R^+\times R^+$ such that $s\leqq t$, $g(t,s,x)$ is continuous in x.

(H3) For each real number $l>0$ and each compact subset K of R^n, there exists a function $m(t,\cdot)\in\mathcal{L}^1[0,t]$ for each $t\in[0,l]$ such that

$$|g(t,s,x)|\leqq m(t,s) \qquad (0\leqq s\leqq t\leqq l,\ x\in K)$$

and

$$\sup\left\{\int_0^t m(t,s)ds : 0\leqq t\leqq l\right\}<\infty.$$

(H4) For each compact subinterval J of R^+, each compact set K in R^n and each t_0 in R^+,

$$\sup\left\{\int_J |g(t,s,\phi(s))-g(t_0,s,\phi(s))|\,ds : \phi\in C(J;K)\right\}$$

tends to zero as $t\to t_0$.

(H5) Given any constant $l>0$ and any compact set $K\subset R^n$, we have

$$\lim_{h\to 0}\int_t^{t+h}|g(t+h,s,\phi(s))|\,ds=0$$

uniformly in (t,ϕ) for $0\leqq t\leqq l$ and $\phi\in C([0,l+1];K)$.

We define approximate solutions, sometimes called Carathèodory iterates, which will be used in the proof of the main theorem in our later paper [3]. A function $x(t;\xi,\varepsilon)$ is said to be an ε-Carathèodory iterate at a point $\xi\in[0,\alpha]$ for a continuous solution $x(t)$ of (P) on $[0,\alpha]$, or simply a Carathèodory iterate, if

$$(1)\qquad x(t;\xi,\varepsilon)=\begin{cases} f(0) & \text{on } [-\varepsilon,0]\\ x(t) & \text{on } [0,\xi]\\ f(t)+\int_0^\xi g(t,s,x(s))ds+\int_\xi^t g(t,s,x(s-\varepsilon;\xi,\varepsilon))ds & \\ & \text{on } [\xi,\alpha].\end{cases}$$

We shall give some explanation of this definition in the following Proposition 1.

Proposition 1. *Let the functions f and g satisfy* (H1)–(H4), *then a Carathèodory iterate $x(t;\xi,\varepsilon)$ is defined and continuous on $[0,\alpha]$ for each $\xi\in[0,\alpha]$ and $\varepsilon>0$.*

Proof. The last term of the formula (1) defines a continuous function $x(t;\xi,\varepsilon)$ for $[\xi,\xi+\varepsilon]$. For if we take a compact set $K_0=\cup\{x(t): 0\leqq t\leqq\xi\}$ and $l=\xi+\varepsilon$ in (H3), then we see that $x(t;\xi,\varepsilon)$ is defined and bounded on $[\xi,\xi+\varepsilon]$ by (H2) and (H3), and that $x(t;\xi,\varepsilon)$ is continuous on $[0,\xi+\varepsilon]$ by (H1), (H2) and (H4), because if $x(t)$ is continuous on $[0,\xi]$ and $t, t+h\in[\xi,\xi+\varepsilon]$ the inequality

$$\begin{aligned}&|x(t+h;\xi,\varepsilon)-x(t;\xi,\varepsilon)|\\ &\quad\leqq|f(t+h)-f(t)|+\int_0^\xi|g(t+h,s,x(s))-g(t,s,x(s))|\,ds\\ &\qquad+\int_\xi^{\xi+\varepsilon}|g(t+h,s,x(s-\varepsilon;\xi,\varepsilon))-g(t,s,x(s-\varepsilon;\xi,\varepsilon))|\,ds\end{aligned}$$

holds. Here we note that $K_1=\cup\{x(t;\xi,\varepsilon): -\varepsilon\leqq t\leqq\xi+\varepsilon\}$ is compact. It then follows that (1) can be used to extend $x(t;\xi,\varepsilon)$ as a continuous function over $[-\varepsilon,\xi+2\varepsilon]$. Continuing in this fashion (1) serves to define $x(t;\xi,\varepsilon)$ over $[0,\alpha]$.

For each positive integer n, define $x_n(t)$ by $x(t)=x_n(t;0,1/n)$. Here, we can give another proof of the existence theorem in [1] by using Carathèodory iterates $\{x_n\}$.

Theorem 1. *Under the hypotheses* (H1)–(H4), *there exists an interval* $[0,\beta]$, $\beta>0$, *on which there is a continuous solution* $x(t)$ *of* (P).

We shall only give a brief sketch of the proof. We can find an interval $[0,\beta]$ and a compact set $K\subset R^n$ such that

$$K=\overline{\cup\{K(t):t\in[0,\beta]\}} \qquad \text{(the closure in } R^n).$$

$$K(t)=\{p\in R^n:|p-f(t)|<\delta\} \quad \text{and}$$

$$\delta=\sup\left\{\int_0^t|g(t,s,\phi(s))|\,ds:0\leqq t\leqq\beta,\ \phi\in C([0,\beta];K)\right\}.$$

Then each approximate solution $x_n(t)$ is defined and continuous on $[0,\beta]$. Moreover $x_n(\cdot)\in D[0,\beta]$, where the set $D[0,\beta]$ is defined by

$$D[0,\beta]=\{x(\cdot)\in C[0,\beta]:x(t)\in K(t) \text{ for every } t\in[0,\beta]\}.$$

Hence we see from (H3) and (H4) that the sequence $\{x_n\}$ is equi-continuous and uniformly bounded on $[0,\beta]$, and so $\{x_n\}$ has a subsequence with a limit, x say. Then $x(t)$ is a solution of (P) on $[0,\beta]$.

For any $T>0$ we put $F^*(T)=\cup\{F(t):0\leqq t\leqq T\}$, where $F(t)$ is the cross-section $F(t)=\{p:p=x(t),$ where x is some solution of (P)$\}$. Let α_M be the positive number $\alpha_M=\sup\{\beta>0:F^*(\beta)$ is compact$\}$. By (H5) we see that $[0,\alpha_M)$ becomes a right maximal interval (for details, see [2]).

Proposition 2. *Let the Hypotheses* (H1)–(H4) *be satisfied, and let c be a fixed number in* $[0,\alpha_M)$. *Then for any* $r_0>0$, *there exists an* $\varepsilon_0>0$ *such that an ε-Carathèodory iterate* $x(t;\xi,\varepsilon)$ *at* $\xi\in[0,c]$ *for a fixed solution* $x(t)$ *of* (P) *on* $[0,c]$ *belongs to* $V(F^*(c),r_0)$ *for all* $\varepsilon\in(0,\varepsilon_0]$ *and every* $t,\xi\in[0,c]$, *where* $V(F^*(c),r_0)$ *is an r_0-neighbourhood of* $F^*(c)$.

Proof. To prove this proposition assume the contrary. Then without loss of generality we can assume that there exists a sequence of Carathèodory iterates $\{x(\cdot;\xi_n,\varepsilon_n)\}$ such that

(I) $\lim_{n\to\infty}\varepsilon_n=0$ (monotonely decreasing) and $\lim_{n\to\infty}\xi_n=\xi_0$

(II) $x(t;\xi_n,\varepsilon_n)\in V(F^*(c),r_0)$ for $t\in[0,t_n)$ and
$x(t_n;\xi_n,\varepsilon_n)\in\partial V(F^*(c),r_0)$ (the boundary of $V(F^*(c),r_0)$)

(III) $\lim_{n\to\infty}x(t_n;\xi_n,\varepsilon_n)=x_0\in\partial V(F^*(c),r_0)$ and $\lim_{n\to\infty}t_n=t_0$.

We can verify that $0\leqq\xi_0<t_0\leqq c$. Moreover in (I) and in (III), we can assume that the sequences $\{\xi_n\}$ and $\{t_n\}$ converge monotonely (monotonely decreasing or monotonely increasing). Hence we can consider four cases.

Case (A): $\lim_{n\to\infty}\xi_n=\xi_0$, $\lim_{n\to\infty}t_n=t_0$ (monotonely increasing). In this case we define a family of continuous functions $\{\bar{x}(\cdot;\xi_n,\varepsilon_n)\}$ on $[0,t_0]$

as follows:

$$\bar{x}(t\,;\xi_n,\varepsilon_n)=\begin{cases}x(t\,;\xi_n,\varepsilon_n) & \text{on } [0,t_n]\\ x(t_n\,;\xi_n,\varepsilon_n) & \text{on } [t_n,t_0].\end{cases}$$

Then by (II) $\bar{x}(t\,;\xi_n,\varepsilon_n)$ belongs to the closure $\overline{V(F^*(c),r_0)}$ for every $t\in[0,t_0]$, ξ_n and $\varepsilon_n>0$, and therefore the family $\{\bar{x}(\cdot\,;\xi_n,\varepsilon_n)\}$ is uniformly bounded.

Let $t\in[0,\xi_n]$, then

$$\begin{aligned}|\bar{x}(t+h\,;\xi_n,\varepsilon_n)-\bar{x}(t\,;\xi_n,\varepsilon_n)|&=|x(t+h)-x(t)|\\ &\leqq\sup\{|x(t+h)|-x(t)|:t\in[0,t_0]\}\\ &=I_0(h),\qquad \text{if } t+h\in[0,\xi_n].\end{aligned}$$

Let $t\in[\xi_n,t_n]$, then

$$\begin{aligned}|\bar{x}(t+h\,;\xi_n,\varepsilon_n)-\bar{x}(t\,;\xi_n,\varepsilon_n)|&\leqq|f(t+h)-f(t)|\\ &\quad+\int_0^{\xi_n}|g(t+h,s,x(s))-g(t,s,x(s))|\,ds\\ &\quad+\int_{\xi_n}^{t+h}|g(t+h,s,x(s-\varepsilon_n\,;\xi_n,\varepsilon_n))-g(t,s,x(s-\varepsilon_n\,;\xi_n,\varepsilon_n))|\,ds\\ &\leqq\sup\{|f(t+h)-f(t)|:t\in J\}\\ &\quad+2\sup\left\{\int_J|g(t+h,s,\phi(s))-g(t,s,\phi(s))|\,ds\,;\phi\in C(J\,;K)\right\}\\ &=I_1(h)+2I_2(t,h),\end{aligned}$$

if $t+h\in[\xi_n,t_n]$ where $J=[0,t_0]$ and $K=\overline{V(F^*(c),r_0)}$. And let $t\in[t_n,t_0]$, then

$$|\bar{x}(t+h\,;\xi_n,\varepsilon_n)-\bar{x}(t\,;\xi_n,\varepsilon_n)|=0,\quad \text{if}\quad t+h\in[t_n,t_0].$$

We shall now show that $\{\bar{x}(\cdot\,;\xi_n,\varepsilon_n)\}$ is equi-continuous at each point $t\in[0,t_0]$. Let t be fixed. Then we can verify the following inequalities as above:

$$|\bar{x}(t+h\,;\xi,\varepsilon_n)-\bar{x}(t\,;\xi_n,\varepsilon_n)|\leqq\begin{cases}I_0(h) & \text{for } t+h\in[0,\xi_n]\\ I_1(h)+2I_2(t,h) & \text{for } t+h\in[\xi_n,t_n]\\ I_1(t_n-t)+2I_2(t_n,t_n-t) & \text{for } t+h\in[t_n,t_0]\end{cases}$$

for all n satisfying $t\in[0,\xi_n]$,

$$|\bar{x}(t+h\,;\xi_n,\varepsilon_n)-\bar{x}(t\,;\xi_n,\varepsilon_n)|\leqq\begin{cases}I_1(h)+2I_2(t,h) & \text{for } t+h\in[0,t_n]\\ I_1(t_n-t)+2I_2(t_n,t_n-t) & \text{for } t+h\in[t_n,t_0]\end{cases}$$

for all n satisfying $t\in[\xi_n,t_n]$, and

$$|\bar{x}(t+h\,;\xi_n,\varepsilon_n)-\bar{x}(t\,;\xi_n,\varepsilon_n)|\leqq\begin{cases}I_1(t_n-t)+2I_2(t_n,t_n-t) & \text{for } t+h\in[0,t_n]\\ 0 & \text{for } t+h\in[t_n,t_0]\end{cases}$$

for all n satisfying $t\in[t_n,t_0]$. Since f and x are continuous on the compact interval $[0,t_0]$, $\lim_{h\to0}I_0(h)=\lim_{h\to0}I_1(h)=0$. Hence $\lim_{h\to0}I_1(t_n-t)=0$, because $0\leqq t_n-t\leqq h$. Hypothesis (H4) with $J=[0,t_0]$ and $K=\overline{V(F^*(c),r_0)}$ implies $\lim_{h\to0}I_2(t,h)=0$. Moreover, it follows from Hypothesis (H4) that $\lim_{h\to0}I_2(t,h)=0$ uniformly in $t\in J$ by the standard argument on uniform continuity on compact sets. Thus we have $\lim_{h\to0}I_2(t_n,t_n-t)=0$. This shows the equi-continuity of $\{\bar{x}(\cdot\,;\xi_n,\varepsilon_n)\}$ at a point $t\in J$. Hence $\{\bar{x}(\cdot\,;\xi_n,\varepsilon_n)\}$ is relatively compact in $\mathcal{C}(J\,;K)$ by Ascoli-Arzelà's Theorem. Thus we can find a subsequence $\{\bar{x}(\cdot\,;\xi_{n_k},\varepsilon_{n_k})\}\subset\{\bar{x}(\cdot\,;\xi_n,\varepsilon_n)\}$ and $x_0(t)\in\mathcal{C}(J\,;K)$ such that $\lim_{k\to\infty}\bar{x}(t\,;\xi_{n_k},\varepsilon_{n_k})=x_0(t)$ uniforly in $t\in J$. For notational convenience we shall write n for n_k. If we can show that

$$x_0(t) \text{ is a solution of (P) on } J=[0,t_0] \tag{1}$$

and

$$\lim_{n\to\infty}\bar{x}(t_0,\xi_n,\varepsilon_n)=x_0\in\partial V(F^*(c),r_0) \tag{2}$$

then we will have shown that $x(t_0)=x_0$. This result would contradict $x_0\in\partial V(F^*(c),r_0)$ and the proof of our Proposition 2 in Case (A) would be complete. (2) is trivial by the definition of $\bar{x}(t\,;\xi_n,\varepsilon_n)$. We shall now show that $x_0(t)$ is a solution of (P). By our construction, the relation

$$\bar{x}(t\,;\xi_n,\varepsilon_n)=f(t)+\int_0^t g(t,s,\bar{\bar{x}}(s\,;\xi_n,\varepsilon_n))ds$$

holds on $[0,t_n]$ and

$$\bar{x}(t\,;\xi_n,\varepsilon_n)=f(t_n)+\int_0^{t_n} g(t_n,s,\bar{\bar{x}}(s\,;\xi_n,\varepsilon_n))ds$$

on $[t_n,t_0]$, where

$$\bar{\bar{x}}(t\,;\xi_n,\varepsilon_n)=\begin{cases}\bar{x}(t\,;\xi_n,\varepsilon_n) & t\in[0,\xi_n]\\ \bar{x}(t-\varepsilon_n\,;\xi_n,\varepsilon_n) & t\in[\xi_n,t_0].\end{cases}$$

For any fixed $t\in[0,t_0)$, the condition $\lim_{n\to\infty}t_n=t_0$ (monotonely increasing) implies that there exists $N>0$ such that

$$\bar{x}(t\,;\xi_n,\varepsilon_n)=f(t)+\int_0^t g(t,s,\bar{\bar{x}}(s\,;\xi_n,\varepsilon_n))ds$$

for any $n\geqq N$. Here we note that

$$|g(t,s,\bar{\bar{x}}(s\,;\xi_n,\varepsilon_n))|\leqq m(t,s)\qquad(0\leqq s\leqq t\leqq t_0,\ n=1,2,\cdots),$$

where $m(t,\cdot)$ is the measurable function in $\mathcal{L}^1[0,t]$ stated in (H3) corresponding to $l=t_0$ and $K=\overline{V(F^*(c),r_0)}$. By the equi-continuity of $\{\bar{x}(\cdot\,;\xi_n,\varepsilon_n)\}$, we can verify that $\lim_{n\to\infty}\bar{\bar{x}}(t-\varepsilon_n,\xi_n,\varepsilon_n)=x_0(t)$ for every $t\in[0,t_0)$. Therefore by the Lebesgue dominated convergence theorem we have

$$x_0(t)=f(t)+\int_0^t g(t,s,x_0(s))ds.$$

We can show that this equality holds also at $t=t_0$, because by the continuity of $x_0(t)$ and (H4) we have

$$\begin{aligned}x_0(t_0)&=\lim_{t\to t_0}x_0(t)\\&=\lim_{t\to t_0}f(t)+\lim_{t\to t_0}\int_0^t g(t,s,x(s))ds\\&=f(t_0)+\int_0^{t_0}g(t_0,s,x_0(s))ds.\end{aligned}$$

Hence $x_0(t)$ is a solution of (P) on $[0,t_0]$. Thus (1) is verified.

Case (B): $\lim_{n\to\infty} \xi_n = \xi_0$ (monotonely increasing) and $\lim_{n\to\infty} t_n = t_0$ (monotonely decreasing). We define in this case $\bar{x}(t;\xi_n,\varepsilon_n)$ on $[0,t_0]$ by $\bar{x}(t;\xi_n,\varepsilon_n)=x(t;\xi_n,\varepsilon_n)$. Then we can suppose that $\{\bar{x}(\cdot\,;\xi_n,\varepsilon_n)\}$ is an equi-continuous family on $[0,t_0]$ with a uniform limit $x_0(t)$. Then we can prove as before that $x_0(t)$ satisfies (1). Moreover the equi-continuity of $\{x(\cdot\,;\xi_n,\varepsilon_n)\}$ and (III) imply that (2) is also true in this case. Hence, Case (B) can be proved by contradiction as before.

Other cases can be demonstrated in similar fashion.

Remark. In the Proposion 2 above, ε_0 depends on r_0 and $x(\cdot)$. This result can be improved to that ε_0 depends on r_0 only, if we use instead of $x(\cdot\,;\xi_n,\varepsilon_n)$ new Carathèodory iterates $x_n(\cdot\,;\xi_n,\varepsilon_n)$ constructed from a sequence of solutions $\{x_n(\cdot)\}$ with a uniform limit $x(\cdot)$.

About the continuity in ξ of $x(t;\xi,\varepsilon)$, we have the following theorem.

Theorem 2. *Let f and g satisfy the conditions of Proposition* 2. *Then for any solution x of* (P) *and every $\varepsilon>0$, Carathèodory iterates $x(\cdot\,;\xi,\varepsilon)$ belong to $C[0,\alpha_M)$ and $x(\cdot\,;\xi,\varepsilon)$ is continuous in $\xi\in[0,\alpha_M)$ with the compact-open topology of $C[0,\alpha_M)$.*

The proof of this theorem will be found in our forecoming note [3].

References

[1] R. K. Miller and G. R. Sell: Existence, uniqueness and continuity of solutions of integral equations. An addendum. Ann. Mat. Pura. Appl., **87**, 281–286 (1970).

[2] R. K. Miller: Nonlinear Volterra Integral Equations. Mathematical Lecture Note Series, Benjamin (1971).

[3] S. Nakagiri and H. Murakami: Kneser's property of solution families of non-linear volterra integral equations (to appear).