47. *Complex Hypersurfaces with Vanishing Bochner Curvature Tensor*

By Minoru KOBAYASHI
Department of Mathematics, Faculty of Science, Josai University, Saitama

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1. Introduction. The purpose of this paper is to show the following:

Theorem. *Let M be a complex hypersurface of complex dimension n $(n\geqq 2)$ in a space of constant holomorphic sectional curvature $\tilde{c}$. If the Bochner curvature tensor of M vanishes identically, then M is of constant holomorphic sectional curvature $\tilde{c}$.*

2. Preliminaries. Let $(\tilde{M}, J, g)$ be a Kaehlerian manifold of constant holomorphic sectional curvature $\tilde{c}$ of complex dimension $n+1$ $(n\geqq 2)$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}$ is given by

$$(1)\qquad \tilde{R}(\tilde{X},\tilde{Y})\tilde{W}=\frac{\tilde{c}}{4}\{g(\tilde{Y},\tilde{W})\tilde{X}-g(\tilde{X},\tilde{W})\tilde{Y}+g(J\tilde{Y},\tilde{W})J\tilde{X} -g(J\tilde{X},\tilde{W})J\tilde{Y}+2g(\tilde{X},J\tilde{Y})J\tilde{W}\},$$

where $\tilde{X}$, $\tilde{Y}$ and $\tilde{W}$ are vector fields on $\tilde{M}$. Let M be a complex hypersurface of $\tilde{M}$ immersed by $\varphi: M\to\tilde{M}$ and ξ a local field of unit vectors normal to M. Then, identifying, for each $x\in M$, the tangent space $T_x(M)$ with $\varphi_*(T_x(M))\subset T_{\varphi(x)}(\tilde{M})$ by means of φ_*, we may put

$$(2)\qquad \tilde{\nabla}_X\xi=-AX+s(X)J\xi,$$

$$(3)\qquad \tilde{\nabla}_XY=\nabla_XY+h(X,Y)\xi+k(X,Y)J\xi,$$

where $\tilde{\nabla}$ denotes the covariant differentiation with respect to g, X and Y are vector fields in M and $-AX$ (resp. ∇_XY) is the tangential part of $\tilde{\nabla}_X\xi$ (resp. $\tilde{\nabla}_XY$). It is well known that the naturally induced metric is the Kaehlerian metric and the almost complex structure is the Kaehlerian structure on M. We denote them also by g and J respectively. Then the relations $h(X,Y)=g(AX,Y)$, $k(X,Y)=g(JAX,Y)$ and $JA=-AJ$ hold (for details, see [3]).

The curvature tensor R and the Ricci tensor S of M are given by

$$(4)\qquad R(X,Y)=\tilde{R}(X,Y)+AX\wedge AY+JAX\wedge JAY,$$

$$(5)\qquad S(X,Y)=-2g(A^2X,Y)+\frac{n+1}{2}\tilde{c}g(X,Y),$$

where $X\wedge Y$ is the endomorphism defined by $(X\wedge Y)(Z)=g(Z,Y)X-g(X,Z)Y$. The Bochner curvature tensor B of M is, by definition, given by

$$
(6)\quad \begin{aligned} B(X,Y)=R(X,Y)-\frac{1}{2n+4}[R^1X\wedge Y+X\wedge R^1Y+R^1JX\wedge JY \\ +JX\wedge R^1JY-2g(JX,R^1Y)J-2g(JX,Y)R^1\circ J] \\ +\frac{\operatorname{trace}R^1}{(2n+4)(2n+2)}[X\wedge Y+JX\wedge JY-2g(JX,Y)J], \end{aligned}
$$

where $g(R^1X, Y)=S(X, Y)$. We assume the Bochner curvature tensor $B=0$. Then we have, from (6),

$$
(7)\quad \begin{aligned} R(X,Y)W=\frac{1}{2n+4}\{g(W,Y)R^1X-g(R^1X,W)Y+g(W,R^1Y)X \\ -g(X,W)R^1Y+g(W,JY)R^1JX-g(R^1JX,W)JY \\ +g(W,R^1JY)JX-g(JX,W)R^1JY-2g(JX,R^1Y)JW \\ -2g(JX,Y)R^1JW\}-\frac{\operatorname{trace}R^1}{(2n+4)(2n+2)} \\ \{g(W,Y)X-g(X,W)Y+g(W,JY)JX \\ -g(JX,W)JY-2g(JX,Y)JW\}. \end{aligned}
$$

Substituting (1) and (7) into (4), we have

$$
(8)\quad \begin{aligned} \frac{1}{2n+4}\{g(W,Y)R^1X-g(R^1X,W)Y+g(W,R^1Y)X-g(X,W)R^1Y \\ +g(W,JY)R^1JX-g(R^1JX,W)JY+g(W,R^1JY)JX \\ -g(JX,W)R^1JY-2g(JX,R^1Y)JW-2g(JX,Y)R^1JW \\ -\frac{\operatorname{trace}R^1}{(2n+4)(2n+2)}\{g(W,Y)X-g(X,W)Y+g(W,JY)JX \\ -g(JX,W)JY-2g(JX,Y)JW\} \\ =\frac{\tilde{c}}{4}\{g(Y,W)X-g(X,W)Y+g(JY,W)JX-g(JX,W)JY \\ +2g(X,JY)JW\}+g(AY,W)AX-g(AX,W)AY \\ +g(JAY,W)JAX-g(JAX,W)JAY. \end{aligned}
$$

Substituting $R^1X=-2A^2X+\frac{n+1}{2}\tilde{c}X$ into (8), we have, after simplification,

$$
(9)\quad \begin{aligned} \left(\frac{\tilde{c}}{4}+\frac{\operatorname{trace}R^1}{(2n+4)(2n+2)}-\frac{n+1}{2n+4}\tilde{c}\right)\{g(Y,W)X-g(X,W)Y \\ +g(JY,W)JX-g(JX,W)JY+2g(X,JY)JW\} \\ =\frac{1}{2n+4}\{-2g(W,Y)A^2X+2g(A^2X,W)Y-2g(W,A^2Y)X \\ +2g(X,W)A^2Y-2g(W,JY)A^2JX+2g(A^2JX,W)JY \\ -2g(W,A^2JY)JX+2g(JX,W)A^2JY+4g(JX,A^2Y)JW \\ +4g(JX,Y)A^2JW\}-g(AY,W)AX+g(AX,W)AY \\ -g(JAY,W)JAX+g(JAX,W)JAY. \end{aligned}
$$

3. Proof of Theorem. We take an orthonormal basis $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ of $T_x(M)$ such that $Ae_i=\lambda_i e_i$, $AJe_i=-\lambda_i Je_i$, $g(Je_i, e_j)=0$ $(i, j=1, \cdots, n)$ and $0\leqq\lambda_1\leqq\cdots\leqq\lambda_n$. Then, since $\operatorname{trace}R^1=-2\operatorname{trace}A^2$

$+(n+1)n\tilde{c}$, (9) reduces to

$$
\begin{aligned}
&-\frac{\operatorname{trace} A^2}{2(n+1)(n+2)}\{g(Y, W)X-g(X, W)Y+g(JY, W)JX\\
&\qquad -g(JX, W)JY+2g(X, JY)JW\}\\
&=\frac{1}{2n+4}\{-2g(W, Y)A^2X+2g(A^2X, W)Y-2g(W, A^2Y)X\\
&\qquad +2g(X, W)A^2Y-2g(W, JY)A^2JX+2g(A^2JX, W)JY\\
&\qquad -2g(W, A^2JY)JX+2g(JX, W)A^2JY+4g(JX, A^2Y)JW\\
&\qquad +4g(JX, Y)A^2JW\}-g(AY, W)AX+g(AX, W)AY\\
&\qquad -g(JAY, W)JAX+g(JAX, W)JAY. \qquad (10)
\end{aligned}
$$

Putting $X=e_i$, $Y=e_j$, $W=e_k$ in (10), we have

$$
\begin{aligned}
&-\frac{\operatorname{trace} A^2}{2(n+1)(n+2)}(\delta_{jk}e_i-\delta_{ik}e_j)\\
&=\frac{1}{2n+4}(-2\delta_{jk}\lambda_i^2e_i+2\delta_{ik}\lambda_i^2e_j-2\delta_{kj}\lambda_j^2e_i+2\delta_{ik}\lambda_j^2e_j)\\
&\qquad -\lambda_i\lambda_j\delta_{jk}e_i+\lambda_i\lambda_j\delta_{ik}e_j.
\end{aligned}
$$

Setting $j=k\neq i$, we have

$$
\frac{\operatorname{trace} A^2}{2(n+1)(n+2)}=\frac{1}{n+2}(\lambda_i^2+\lambda_j^2)+\lambda_i\lambda_j \qquad (i\neq j), \tag{11}
$$

from which we have

$$
(\lambda_k-\lambda_j)\{\lambda_k+\lambda_j+(n+2)\lambda_i\}=0 \qquad (n\geqq 3)
$$

(for distinct i, j and k).

Therefore the rank A must be 0, $2n-2$ or $2n$. But, if the rank $A\neq 0$, then the non-zero λ_i's must be equal to, say, λ, because of $0\leqq\lambda_1\leqq\cdots\leqq\lambda_n$. Hence, again by (11), we have

$$
\frac{n\lambda^2}{(n+1)(n+2)}=\frac{2\lambda^2}{n+2}+\lambda^2
$$

(if the rank $A=2n$),

$$
\begin{cases}
\dfrac{(n-1)\lambda^2}{(n+1)(n+2)}=\dfrac{2\lambda^2}{n+2}+\lambda^2\\[2mm]
\dfrac{(n-1)\lambda^2}{(n+1)(n+2)}=\dfrac{\lambda^2}{n+2}
\end{cases}
$$

(if the rank $A=2n-2$),

from which we have $\lambda=0$ in both cases. Hence M is totally geodesic. Thus M is of constant holomorphic sectional curvature $\tilde{c}$. If $n=2$, then we have, from (11),

$$
\frac{\lambda_1^2+\lambda_2^2}{12}=\frac{1}{4}(\lambda_1^2+\lambda_2^2)+\lambda_1\lambda_2,
$$

from which we have $\lambda_1=\lambda_2=0$. Hence M is of constant holomorphic sectional curvature $\tilde{c}$, which completes the proof.

References

[1] S. Kobayashi and K. Nomizu: Foundations of Differential Geometry, Vols. I, II. Intersci. Publ., 1963, 1969.
[2] H. Takagi and Y. Watanabe: Kählerian manifolds with vanishing Bochner curvature tensor satisfying $R(X, Y)\cdot R=0$. Hokkaido Math. Jour. Vol. III, 129–132 (1974).
[3] B. Smyth: Differential geometry of complex hypersurfaces. Ann. of Math., **85**, 246–266 (1967).