45. *A Note on Hypercommutativity of Operators in Real Banach Spaces*

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In [1], C. Apostol generalized to a complex Banach space an invariant subspace theorem of C. Pearcy and N. Salinas in a complex Hilbert space [8]. In a finite dimensional complex vector space every linear operator has at least one eigenvector (one-dimensional invariant subspace). This result which played a fundamental role in the development of the theory of complex vector spaces does not apply in the case of real spaces. The purpose of this note is to show the corresponding assertion of [1] in real Banach spaces. We base our arguments on C. Apostol's paper [1].

In this note, X will denote a separable Banach space over R (the set of all real numbers) of dimension greater than two, $\mathscr{B}(X)$ the algebra of all bounded linear operators acting in X, $\mathscr{E}(X)$ the set of all finite dimensional subspaces of X. If M is a non-empty subset of X and $x \in X$, the *distance* from x to M, $d(x, M)$, is defined by $d(x, M) = \inf \{\|x-y\| : y \in M\}$. In the sequel, a *subspace* means a closed linear manifold.

Definition 1 ([5]). Given a sequence $\{X_n\}$ of subspaces of X, define $\liminf X_n$ to be

$$\liminf X_n = \{x \in X : \lim d(x, X_n) = 0\}.$$

It is clear that $\liminf X_n$ is a subspace of X and $\liminf X_{n_k} = \liminf X_n$ for any subsequence $\{n_k\}$ of $\{n\}$. If for every $n \geqq 1$, X_n is a subspace of Y_n, then $\liminf X_n \subset \liminf Y_n$.

Definition 2. Let $\mathscr{A}$ be a set of operators, $\mathscr{A} \subset \mathscr{B}(X)$. Then an *invariant subspace for* $\mathscr{A}$ is an invariant subspace for all operators in $\mathscr{A}$.

Definition 3. Let $\{X_n\} \subset \mathscr{E}(X)$ and $\mathcal{C}_n \subset \mathscr{B}(X_n)$. We define $\liminf \mathcal{C}_n$ to be

$$\liminf \mathcal{C}_n = \left\{T \in \mathscr{B}(X) : \lim \left(\inf_{S_n \in \mathcal{C}_n} \|T | X_n - S_n\|\right) = 0\right\}.$$

It is clear that $\liminf \mathcal{C}_{n_k} = \liminf \mathcal{C}_n$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\mathcal{C}_n, \mathscr{D}_n \subset \mathscr{B}(X_n)$. If for every $n \geqq 1$, $\mathcal{C}_n \subset \mathscr{D}_n$, then $\liminf \mathcal{C}_n \subset \liminf \mathscr{D}_n$.

Lemma 1 (P. Meyer-Nieberg [6] or [5]). *Let* $\{X_n\}$ *and* $\{Y_n\}$ *be two* *sequences of subspaces of* X *such that* $\liminf X_{n_k}=\liminf X_n$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$ *and* $X_n\subset Y_n$, $\dim (Y_n/X_n)\leqq m$ *for all* n, *then*

$$\dim (\liminf Y_n/\liminf X_n)\leqq m.$$

The following Lemma 2 and Proposition have been proved in [1].

Lemma 2. *Let* $X_n, Y_n\in \mathcal{E}(X)$, $\mathcal{C}_n\subset \mathcal{B}(X_n)$. *If* Y_n *is an invariant subspace for* $\mathcal{C}_n$, *then* $\liminf Y_{n_k}$ *is an invariant subspace for* $\liminf \mathcal{C}_{n_k}$, *for any subsequence* $\{n_k\}$ *of* $\{n\}$. *If* $\mathcal{C}_{n_k}$ *is commutative, then* $\liminf \mathcal{C}_{n_k}$ *is commutative on* $\liminf X_{n_k}$.

Definition 4. An operator $T\in\mathcal{B}(X)$ is called *quasitriangular* if there exists $\{X_n\}\subset\mathcal{E}(X)$ such that $\liminf X_n=X$ and $T\in\liminf \mathcal{B}(X_n)$.

The concept of quasitriangularity is introduced by P. R. Halmos in the complex Hilbert space [4] and by P. Meyer-Nieberg in the case of a Banach space [7].

Definition 5. Let $\mathcal{F}\subset\mathcal{B}(X)$, $\mathcal{F}\neq\emptyset$. We call $\mathcal{F}$ a *hypercommutative set* if there exists $\{X_n\}\subset\mathcal{E}(X)$, with $\mathcal{C}_n$ commutative and $\mathcal{C}_n\subset\mathcal{B}(X_n)$ such that $\liminf X_n=X$ and $\mathcal{F}\subset\liminf \mathcal{C}_n$.

The concept of hypercommutativity is introduced by C. Apostol [1].

Proposition. (a) *If* T *is a quasitriangular operator then* $\{T\}$ *is a hypercommutative set.*

(b) *Let* $\mathcal{F}$ *be a hypercommutative set and denote by* $\mathcal{C}$ *the inverse-closed and uniformly closed algebra generated by* $\mathcal{F}$. *Then* $\mathcal{C}$ *is a hypercommutative set.*

Lemma 3. *Let* $Y\in\mathcal{E}(X)$, $\dim Y=n$, $\mathcal{C}\subset\mathcal{B}(Y)$. *If* $\mathcal{C}$ *is commutative, then there exists a one or two dimensional invariant subspace* (*of* Y) *for* $\mathcal{C}$.

Proof. The proof is by induction on the dimension of Y. In the case of one or two dimensional space ($n=1$ or 2), it is obvious. We assume that it is true for spaces of dimension $<n$ ($n\geqq 3$) and prove it for an n-dimensional space. Let Z denote the space $X\times X$. As the sum of elements (x_1, y_1), $(x_2, y_2)\in Z$ we take the element (x_1+x_2, y_1+y_2). As the scalar product, we define such that

$$(\alpha+i\beta)(x, y)=(\alpha x-\beta y, \alpha x+\beta y)$$

for $(x, y)\in Z$ and $\alpha, \beta\in R$. It is easy to see that Z is now a complex vector space. In this space Z, we define the norm $\|\cdot\|_0$ to be

$$\|(x, y)\|_0=\sup \{|f(x)+if(y)|: f\in X^*, \|f\|\leqq 1\},$$

where X^* denote the dual space of X. Then we have $\|x\|+\|y\|\geqq\|(x, y)\|_0\geqq\max\{\|x\|, \|y\|\}$ for every $x, y\in X$. Now, for each operator $A\in\mathcal{C}$ we consider the operator $\tilde{A}$ on Z by

$$\tilde{A}(x, y)=(Ax, Ay).$$

It is easy to see that $\tilde{A}$ is a bounded linear operator on the complex normed linear space (finite dimension). Let $\tilde{\mathcal{C}}=\{\tilde{A}: A\in\mathcal{C}\}$. Then $\tilde{\mathcal{C}}$ is a commutative set of $\mathcal{B}(Z)$ (by commutativity of $\mathcal{C}$). If every vector of Z is an eigenvector of all operators in $\tilde{C}$, then every vector of X is an eigenvector of all operators in C, in this case our lemma is proved. Assume therefore that there exists a vector in Z which is not an eigenvector of some operator $\tilde{A}$ in $\tilde{C}$. Let $\lambda=\alpha+i\beta$ be one of eigenvalues of $\tilde{A}$ and Z_λ the set of all eigenvectors of $\tilde{A}$ (together with the null vector) corresponding to λ. Since $\tilde{A}\tilde{T}=\tilde{T}\tilde{A}$, we have $\tilde{A}\tilde{T}(x,y)=\tilde{T}\tilde{A}(x,y)=\tilde{T}\lambda(x,y)=\lambda\tilde{T}(x,y)$ for $(x,y)\in Z_\lambda$, i.e. $\tilde{T}(x,y)\in Z_\lambda$. The space Z_λ is an invariant subspace for $\tilde{C}$. We consider X_λ spanned by the vectors $x_1, y_1, x_2, y_2, \cdots$ where $(x_1,y_1),(x_2,y_2),\cdots\in Z_\lambda$. By the construction, the space X_λ is an invariant subspace for C. The space X_λ is a subspace of X different from the null space and the whole space. The space X_λ is of a dimension $\leqq n-1$. Since by assumption our lemma is true for spaces of dimension $<n$, X_λ must contain a one or two dimensional invariant subspace for C. This proved our lemma.

Lemma 4. *Let $Y\in\mathcal{E}(X)$, $\dim Y=n$, $C\subset\mathcal{B}(Y)$. If C is commutative, then there exist subspaces $L_0, L_1, \cdots, L_{m-1}, L_m$ with the following properties;*

(i) $\{0\}=L_0\subset L_1\subset\cdots\subset L_{m-1}\subset L_m=Y$,

(ii) $AL_j\subset L_j$ $(j=0,1,2,\cdots,m)$ *for all* $A\in C$,

(iii) $\dim(L_j/L_{j-1})=1$ *or* 2 $(j=1,2,\cdots,m)$.

Proof. The proof is by induction on the dimension of Y. If $n=1,2$ the statement is obvious. We assume that it is true for spaces of dimension $<n$ and prove it for an n-dimensional space. Consider the adjoint operators $A^*, B^*, \cdots$ $(A,B,\cdots\in C)$ on the dual space Y^*, since they have a one or two dimensional common invariant subspace (by Lemma 3), say W. Let us denote by W^0 the anihilator (in $Y^{**}=Y$) of W, then W^0 is an $n-1$ or $n-2$ dimensional subspace of Y [3, Theorem 1 of §17] and W^0 is invariant for C. Consequently we may consider the operators in C as linear operators on W^0, and by assumption, our lemma is true for spaces of dimension $<n$, we may find L_0 $(=\{0\})$, $L_1, \cdots, L_{m-1}$ $(=W^0)$, satisfying the conditions (i)$\sim$(iii). We set $L_m=Y$. This proved our lemma.

Theorem. *Let $\mathcal{F}(\subset\mathcal{B}(X))$ be a hypercommutative set containing a non-zero compact operator K. Then there exists a proper invariant subspace (of X) for $\mathcal{F}$.*

Proof. Let $X_n\in\mathcal{E}(X)$ and $C_n\subset\mathcal{B}(X_n)$, C_n commutative set such that $\liminf X_n=X$ and $\mathcal{F}\subset\liminf C_n$. Since $KT=TK$ for every $T\in\mathcal{F}$ and $K\neq 0$, we may assume that the null space of K is zero, for otherwise $K^{-1}(0)$ is a proper invariant subspace for $\mathcal{F}$. Therefore there exists $e\in X$, $\|e\|=1$, $Ke\neq 0$, and we can choose α with $0<\alpha<1$ and $\alpha\|K\|<\|Ke\|$. Since $e\in\liminf X_n=X$, we may suppose $d(e,X_n)<\alpha$ for any n. By Lemma 4, there exists a chain of invariant subspaces for $\mathcal{C}_n$ ordered by inclusion;

$$\{0\}=L_0^n \subset L_1^n \subset \cdots \subset L_{m-1}^n \subset L_m^n = X_n,$$
$$\dim (L_j^n / L_{j-1}^n) \leqq 2 \qquad (j=1,2,\cdots,m).$$

We have $d(e, L_0^n)=1>\alpha$, $d(e, L_m^n)<\alpha$. Thus for each n there is a greatest j, j_n say, such that $d(e, L_j^n) \geqq \alpha$. Let $F_n = L_{j_n}^n$, $G_n = L_{j_n+1}^n$. Then $d(e, F_n) \geqq \alpha$, $d(e, G_n) < \alpha$ $(n \geqq 1)$. It follows at once from the first of these inequalities that, for any subsequence $\{n_k\}$ of $\{n\}$, $e \notin \liminf F_{n_k}$. Since $d(e, G_n) < \alpha$, there exists a bounded sequence $\{x_n\} \subset G_n$, i.e. $\|x_n\| < \alpha + \|e\| = \alpha + 1$. Using the compactness of K, we have a subsequence $\{n_k\}$ of $\{n\}$ such that $\lim Kx_{n_k} = x \in X$. We show next, that x belongs to $\liminf G_{n_k}$. Since $\mathcal{F} \subset \liminf \mathcal{C}_n$ and $K \in \mathcal{F}$, there exists a sequence $K_n \in \mathcal{C}_n$ such that $\lim \|K|X_n - K_n\| = 0$, we obtain also

$$\begin{aligned} d(x, G_{n_k}) &\leqq \|x - Kx_{n_k}\| + d(Kx_{n_k}, G_{n_k}) \\ &= d(Kx_{n_k} - K_{n_k}x_{n_k}, G_{n_k}) \\ &\leqq \|K|X_{n_k} - K_{n_k}\| \, \|x_{n_k}\| \to 0 \end{aligned}$$

as $k \to \infty$, which means $x \in \liminf G_{n_k}$. Now, on the other hand, $\|Ke - x\| = \lim \|Ke - Kx_{n_k}\| \leqq \alpha \|K\| < \|Ke\|$. Thus we have $x \neq 0$, and so $\liminf G_{n_k}$ will be a proper invariant subspace for $\mathcal{F}$ unless $\liminf G_{n_k} = X$ (by Lemma 2). Since $\liminf F_{n_l} \neq X$ for every subsequence $\{n_l\}$ of $\{n\}$. Now, if $\liminf F_{n_l} = \{0\}$ for every subsequence $\{n_l\}$ of $\{n\}$, by Lemma 1, dim ($\liminf G_{n_k}$) $\leqq 2$. Therefore $\liminf G_{n_k} \neq X$. This commpletes the proof of our theorem.

As an immediate consequence of Theorem and Proposition, we have the following corollary.

Corollary. *Let T be a quasitriangular operator acting in X and denote by $\mathcal{F}$ the inverse-closed, uniformly closed algebra generated by T. If $\mathcal{F}$ contains a compact operator, $K \neq 0$, then $\mathcal{F}$ has a proper invariant subspace in X.*

References

[1] C. Apostol: Hypercommutativity and invariant subspaces. Rev. Roum. Math. Pures et Appl., **17**, 335–339 (1972).

[2] I. M. Gel'fand: Lectures on Linear Algebra. Interscience, New York (1961).

[3] P. R. Halmos: Finite Dimensional Vector Spaces (2nd ed). Van Nostrand, Princeton (1958).

[4] ——: Quasitriangular operators. Acta Sci. Math. (Szeged), **29**, 283–293 (1968).

[5] K. Kitano: A note on invariant subspaces. Tôhoku Math. J., **21**, 144–151 (1969).

[6] P. Meyer-Nieberg: Invariante Unterräume von polynomkompakten Operatoren. Arch. Math., **19**, 180–182 (1968).

[7] ——: Quasitriangulierbare Operatoren und invariante Untervektorräume stetiger linearer Operatoren. Arch. Math., **22**, 186–199 (1971).

[8] C. Pearcy and N. Salinas: An invariant subspace theorem. Michigan Math. J., **20**, 21–31 (1973).