43. *Elliptic Boundary Problems in Non-Compact Manifolds. I*

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1. Introduction. We consider boundary problems for elliptic differential equations. When the manifold (with boundary) is compact and its boundary is smooth, the indexes of elliptic boundary problems are finite (see [1], etc.). When the manifold is not compact or the data are not given on the entire boundary, the situation is different. Such cases will be studied in this and the forthcoming papers (see Theorem 3).

Let $\mathfrak{M}$ be a σ-compact C^∞ manifold (without boundary), and Ω an open subset of $\mathfrak{M}$. Let ω be an open subset of the topological boundary of Ω in $\mathfrak{M}$. Then we denote by Ω^* the pair of Ω and ω. We also use Ω^* to denote the union of Ω and ω. Take an open subset Ω_0 of $\mathfrak{M}$ such that Ω is contained in Ω_0 and the intersection of Ω_0 and the boundary of Ω in $\mathfrak{M}$ is equal to ω.

Let $\mathcal{F}(\Omega_0)$ be a subspace of $\mathcal{D}'(\Omega_0)$ with a locally convex topology (For the notation of our function spaces, see [1].). Let ρ be the restriction mapping of $\mathcal{D}'(\Omega_0)$ to $\mathcal{D}'(\Omega)$. Then we denote by $\mathcal{F}(\Omega^*)$ the space $\rho(\mathcal{F}(\Omega_0))$, that is, $\mathcal{F}(\Omega^*)=\{u \in \mathcal{D}'(\Omega)\,;\, u=\rho(u_0) \text{ for some } u_0 \in \mathcal{F}(\Omega_0)\}$. This space is endowed with the strongest locally convex topology such that ρ is continuous from $\mathcal{F}(\Omega_0)$ onto $\mathcal{F}(\Omega^*)$. Next we denote by $\mathring{\mathcal{F}}(\Omega^*)$ the closed subspace of $\mathcal{F}(\Omega_0)$ defined by $\mathring{\mathcal{F}}(\Omega^*)=\{u \in \mathcal{F}(\Omega_0)\,;\, \operatorname{supp} u \subset \Omega^*\}$.

In this paper we assume that ω is of C^∞ class. Let R denote the trace operator of $C^\infty(\Omega^*)$ onto $C^\infty(\omega)$. Take a C^∞ vector field ν in a neighborhood of ω which is not tangential to ω. By D_ν we denote the differentiation in the direction ν. Write $\gamma_m=(R, R\circ D_\nu, R\circ D_\nu^2, \cdots, R\circ D_\nu^{m-1})$, for a natural number m.

2. Function spaces $C^\infty(\Omega^*)$ and $\mathring{C}^\infty(\Omega^*)$. Proposition 1. *The space $C^\infty(\Omega^*)$ is separable Fréchet Montel and its dual space is isomorphic to $\mathring{\mathcal{E}}'(\Omega^*)$.*

Outline of the proof. Since $C^\infty(\Omega_0)$ is a Fréchet-Schwartz space, $C^\infty(\Omega^*)$ is also Fréchet-Schwartz (see [2]). Then the former part of the proposition follows. Moreover the dual space of $C^\infty(\Omega^*)$ is isomorphic to the polar of $C^\infty(\Omega^*)$ in $\mathcal{E}'(\Omega_0)$. Using a result due to Schwartz [5], p. 93, we can easily obtain the latter part of the proposition.

Proposition 2. *Let $s \in \boldsymbol{R}$ and $\chi \in C^\infty(\Omega_0)$. Take a compact subset K_1 in the interior of $K=\operatorname{supp}\chi$. Set*

$$p(\phi)=\inf\{\|\chi\cdot\psi\|_{(s)}\,;\ \psi\in C^{\infty}(\Omega_0)\ \text{and}\ \psi|_{\Omega}=\phi\},\qquad \phi\in C^{\infty}(\Omega^*),$$

where $\|\cdot\|_{(s)}$ *is a Sobolev norm (see* [1]). *Then* $u\in H^c_{(-s)}(\Omega_0)$ *and* $\operatorname{supp} u \subset\Omega^*\cap K_1$ *implies the existence of a positive constant* α *such that* $\|u\|_p =\inf\{C>0\,;\ |u(\phi)|\leqq C\cdot p(\phi),\ \phi\in C^{\infty}(\Omega^*)\}\leqq\alpha\cdot\|u\|_{(-s)}$. *Moreover* $u\in\mathring{\mathcal{E}}'(\Omega^*)$ *and* $\|u\|_p<\infty$ *implies that* $\operatorname{supp} u\subset\Omega^*\cap K$ *and there exists a positive constant* β *such that* $\|u\|_{(-s)}\leqq\beta\cdot\|u\|_p$.

The proof of this proposition is straightforward and does not contain any difficulty.

We can use the above propositions to apply our result in the previous paper [3], and then we obtain the following results.

3. Elliptic equations in $\mathring{C}^{\infty}(\Omega^*)$. Let P be an $m=2l$-th order elliptic differential operator in Ω_0, that is, $P_m(x,\xi)\neq 0$ if $x\in\Omega_0$ and $\xi\neq 0$, where P_m is the top symbol of P. We consider the following linear differential equation:

$$P(u)=f, \tag{1}$$

where f and u are C^{∞} functions in $\mathring{C}^{\infty}(\Omega^*)$.

Theorem 3. *Suppose Ω_0 is a real analytic manifold and P is an elliptic differential operator with real analytic coefficients in Ω_0. To every relatively compact open subset U of Ω_0 the union of all compact connected components of $\Omega^*\setminus U$ is supposed to be relatively compact. Then the equation* (1) *has a solution $u\in\mathring{C}^{\infty}(\Omega^*)$ for every $f\in\mathring{C}^{\infty}(\Omega^*)$, which satisfies $\phi(f)=0$ when $\phi\in\mathcal{E}'(\Omega^*)$ and ${}^tP(\phi)=0$. Here ${}^tP$ is the dual operator of P.*

Outline of the proof. To prove this theorem, we consider a linear operator T of $C^{\infty}(\Omega^*)$ into $C^{\infty}(\Omega^*)\times C^{\infty}(\omega)^m$ which is defined by $T(u) =(P(u),\gamma_m(u))$. Suppose that the range of T is closed. Let $f\in\mathring{C}^{\infty}(\Omega^*)$ satisfy $\phi(f)=0$ when $\phi\in\mathcal{E}'(\Omega^*)$ and ${}^tP(\phi)=0$. Then $(f,0,\cdots,0)$ is contained in the range of T, and hence there exists $u\in C^{\infty}(\Omega^*)$ which satisfies $P(u)=f$ in Ω and $\gamma_m(u)=0$. Then u becomes a solution of (1). Therefore it is enough to prove that the range of T is closed.

Combining the results of section 2 and the previous paper [3], and the well-known regularity theorem for elliptic boundary problems (see [1]), we immediately obtain the following lemma.

Lemma 4. *The range of T is closed if and only if the following two requirements hold.*

(i) *To every compact set $K\subset\Omega^*$ there exists another compact set $K'\subset\Omega^*$ such that $\Phi\in\mathring{H}^c_{(0)}(\Omega^*)$, $\phi_j\in H^c_{(-m+j+\frac{1}{2})}(\omega)$, $j=1,2,\cdots,m$, and $\operatorname{supp}({}^tP(\Phi)+\sum_{j=1}^m {}^tD_\nu^{j-1}\circ{}^tR(\phi_j))\subset K$ implies the existence of another $\Psi\in\mathring{H}^c_{(0)}(\Omega^*)$ and $\psi_j\in H^c_{(-m+j+\frac{1}{2})}(\omega)$, $j=1,2,\cdots,m$ which satisfy*

$$\operatorname{supp}\Psi\subset K',\quad \operatorname{supp}\psi_j\subset K'\cap\omega,\qquad j=1,2,\cdots,m, \tag{2}$$

and

$$ {}^tP(\Phi)+\textstyle\sum_{j=1}^m {}^tD_\nu^{j-1}\circ{}^tR(\phi_j)={}^tP(\Psi)+\sum_{j=1}^m {}^tD_\nu^{j-1}\circ{}^tR(\psi_j). \tag{3}$$

(ii) *To every compact set $K \subset \Omega^*$ there exists a positive constant C such that $\Phi \in \mathring{H}^c_{(0)}(\Omega^*)$, $\phi_j \in H^c_{(-m+j+\frac{1}{2})}(\omega)$,* supp $\Phi \subset K$, *and* supp $\phi_j \subset K \cap \omega$, *$j=1, 2, \cdots, m$ implies the existence of another $\Psi \in \mathring{H}^c_{(0)}(\Omega^*)$, and $\psi_j \in H^c_{(-m+j+\frac{1}{2})}(\omega)$, $j=1, 2, \cdots, m$ which satisfy* (2), (3), *and*

$$(4)\qquad \|\Psi\|_{(0)} + \sum_{j=1}^m \|\psi_j\|_{(-m+j+\frac{1}{2})} \leqq C \cdot \|{}^tP(\Psi) + \sum_{j=1}^m {}^tD_\nu^{j-1} \circ {}^tR(\psi_j)\|_{(-m)}.$$

Now we prove the requirement (i) of the above Lemma. Let K be a compact subset of Ω^*. Choose a relatively compact open subset U of Ω_0 which contains K. Let K' denote the closure of the union of $\bar{U} \cap \Omega^*$ and all compact connected components of $\Omega^* \backslash U$. From an assumption of Theorem 3, K' is a compact subset of Ω^*.

Suppose that $\Phi \in \mathring{H}^c_{(0)}(\Omega^*)$, $\phi = (\phi_1, \cdots, \phi_m) \in \mathcal{F} = \prod_{j=1}^m H^c_{(-m+j+\frac{1}{2})}(\omega)$ and supp $({}^tP(\Phi) + {}^t\gamma_m(\phi)) \subset K$. Then ${}^tP(\Phi)$ is equal to zero in $\Omega \backslash K$, and hence Φ is real analytic in $\Omega \backslash K$ (see Petrowsky [4]). Therefore Φ is equal to zero in $\Omega \backslash K'$. Since Φ is an L^2 function with support in Ω^*, Φ is equal to zero in $\Omega_0 \backslash K'$ Then ${}^t\gamma_m(\phi)$ is also equal to zero in $\omega \backslash K'$ and hence ϕ becomes zero in $\omega \backslash K'$. This completes the proof of (i).

Next we prove the requirement (ii) of Lemma 4. Let K be a compact subset of Ω^*. Suppose that $\Phi \in \mathring{H}^c_{(0)}(\Omega^*)$, $\phi = (\phi_1, \cdots, \phi_m) \in \mathcal{F}$, supp $\Phi \subset K$, and supp $\phi_j \subset K \cap \omega$, $j=1, \cdots, m$. Write $\|\phi\|'_{(s)} = \sum_{j=1}^m \|\phi_j\|_{(s+j+\frac{1}{2})}$, $s \in \boldsymbol{R}$. In the following C represents a generic constant which does not depend on the choice of Φ and ϕ. Since ${}^tP$ is elliptic and hence has a parametrix, we have the following estimate.

$$(5)\qquad \|\Phi\|_{(0)} \leqq C \cdot \|{}^tP(\Phi)\|_{(-m)} + C \cdot \|\Phi\|_{(-1)}.$$

Let χ be a C^∞ function with compact support in ω such that $\chi = 1$ in a neighborhood of $K \cap \omega$. Then we can prove the existence of a continuous linear operator S of $\mathcal{G} = \prod_{j=1}^m H^{\mathrm{loc}}_{(m-j-\frac{1}{2})}(\omega)$ into $H^{\mathrm{loc}}_{(m)}(\Omega^*)$ and a pseudo-differential operator $Q_{-\infty}$ of degree $-\infty$ on ω such that $P \circ S(\chi \cdot u) = 0$ and $T \circ S(\chi \cdot u) = \chi \cdot u + Q_{-\infty}(\chi \cdot u)$, $u \in \mathcal{G}$ (see [6], etc.). Using these S and $Q_{-\infty}$, we can easily obtain the estimate

$$(6)\qquad \|\phi\|'_{(-m)} \leqq C \cdot \|{}^tP(\Phi) + {}^t\gamma_m(\phi)\|_{(-m)} + C \cdot \|\phi\|'_{(-m-1)}.$$

From (5) and (6) we have the estimate

$$(7)\qquad \|\Phi\|_{(0)} + \|\phi\|'_{(-m)} \leqq C \cdot \|{}^tP(\Phi) + {}^t\gamma_m(\phi)\|_{(-m)} + C \cdot \|\Phi\|_{(-1)} + C \cdot \|\phi\|'_{(-m-1)}.$$

Now suppose that (ii) of Lemma 4 does not hold. Then there exist sequences $\Phi_n \in \mathring{H}^c_{(0)}(\Omega^*)$ and $\phi_{(n)} \in \mathcal{F}$, $n=1, 2, \cdots$ such that

(a) supp $\Phi_n \subset K$, supp $\phi_{(n)} \subset K \cap \omega$, and $\|{}^tP(\Phi_n) + {}^t\gamma_m(\phi_{(n)})\|_{(-m)} \to 0$ as $n \to \infty$,

(b) there exist $\Psi_n \in \mathring{H}^c_{(0)}(\Omega^*)$ and $\psi_{(n)} \in \mathcal{F}$ which satisfy supp $\Psi_n \subset K$, supp $\psi_{(n)} \subset K \cap \omega$, ${}^tP(\Psi_n) + {}^t\gamma_m(\psi_{(n)}) = {}^tP(\Phi_n) + {}^t\gamma_m(\phi_{(n)})$, and $\|\Psi_n\|_{(0)} + \|\psi_{(n)}\|'_{(-m)} = 1$, and

(c) $\Psi \in \mathring{H}^c_{(0)}(\Omega^*)$, $\psi \in \mathcal{F}$, supp $\Psi \subset K$, supp $\psi \subset K \cap \omega$, and ${}^tP(\Psi) + {}^t\gamma_m(\psi) = {}^tP(\Phi_n) + {}^t\gamma_m(\phi_{(n)})$ implies $\|\Psi\|_{(0)} + \|\psi\|'_{(-m)} \geqq 1$.

From Rellich's theorem there exist subsequences of Ψ_n and $\psi_{(n)}$,

$n=1,2,\cdots$ which converges to some Ψ_0 and $\psi_{(0)}$ with respect to norms $\|\cdot\|_{(-1)}$ and $\|\cdot\|'_{(-m-1)}$ respectively. We write the subsequences by the same letters. Then ${}^tP(\Psi_n)+{}^t\gamma_m(\psi_{(n)})$ converges to ${}^tP(\Psi_0)+{}^t\gamma_m(\psi_{(0)})=0$. Set $\Psi'_n=\Psi_n-\Psi_0$ and $\psi'_{(n)}=\psi_{(n)}-\psi_{(0)}$, $n=1,2,\cdots$. Then it follows from (c) that $\|\Psi'_n\|_{(0)}+\|\psi'_{(n)}\|'_{(-m)}\geqq 1$. But this contradicts (7), and hence (ii) holds. This completes the proof of Theorem 3.

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