41. The Asymptotic Distribution of the Lower Part Eigenvalues for Elliptic Operators

By Hideo TAMURA
Department of Mathematics, Nagoya University

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1. Introduction. Let A be a positive homogeneous elliptic operator with constant coefficients defined on R^n. We consider the eigenvalue problem of the following form

(1.1) $$Au-pu=\lambda u.$$

Here $p(x)$ is a positive function with $\lim_{|x|\to\infty} p(x)=0$. If $p(x)$ does not approach to zero too rapidly at infinity, then the operator $A-p$ has an infinite sequence of negative eigenvalues approaching to zero. We denote by $n(r)$ $(r>0)$ the number of eigenvalues less than $-r$ of problem (1.1). In this note we study the asymptotic behavior of $n(r)$ as $r\to 0$. The asymptotic behavior for the Schrödinger operator with a non-smooth potential $p(x)$ was studied in Brownell and Clark [3], and McLeod [4].

Only the theorem and a sketch of its proof are presented here and the details will be published elsewhere.

2. Main result. Let $A(D)=\sum_{|\alpha|=m} a_\alpha D^\alpha$ be an elliptic operator with constant coefficients defined on R^n. We suppose that $A(\xi)\geq 0$ and denote by $K(l,a)$ $(l>0, a>0)$ the set of functions $p(x)$ which satisfy the following conditions:

(i) $p(x)$ is decomposed as $p(x)=p_1(x)+p_2(x)$;

(ii) $p_1(x)$ is a positive smooth function with $\lim_{|x|\to\infty} |x|^l p_1(x)=a$;

(iii) $p_2(x)$ is a nonnegative function with compact support;

(iv) $p_2(x)\in L_p$, where $p=1$ if $m\geq n$ and $p>n/m$ if $m<n$.

Theorem. *Let A be an elliptic operator satisfying the above conditions and suppose that $p(x)$ belongs to $K(l,a)$ and that $l<m$. Then,*

(2.1) $$n(r)=(2\pi)^{-n}\omega\frac{S}{n}a^{n/l}r^{n/m-n/l}+o(r^{n/m-n/l})$$

where $\omega=\int_{R^n}\frac{d\xi}{(A(\xi)+1)^{n/l}}$ *and S is the surface measure of the $n-1$ dimensional unit sphere if $n\geq 2$ and $S=2$ if $n=1$.*

Remark. Theorem 1 can be extended to the case that $A(D)$ is an inhomogeneous elliptic operator. The details will be discussed in the forthcoming paper.

3. Outline of the proof. In Birman [1], it was shown that $n(r)$ coincides with the number of eigenvalues μ less than 1 of the following eigenvalue problem

(3.1) $$Au+ru=\mu pu.$$

Here we put $r=1/\lambda$ ($\lambda\to\infty$) and consider the eigenvalue problem

(3.2) $$\lambda Au+u=hpu.$$

We denote by $N_\lambda(h)$ the number of eigenvalues less than h of problem (3.2). Obviously $n(r)=N_\lambda(\lambda)$.

For the sake of simplicity, only the case that $m>l>n/2$ is considered. Firstly we suppose that $p(x)=p_1(x)$. Problem (3.2) is transformed to the equivalent eigenvalue problem of the following form

(3.3) $$p^{-\frac{1}{2}}(\lambda A+1)p^{-\frac{1}{2}}v=hv.$$

We denote by $\{\mu_j>0\}_{j=1}^\infty$ and $\{\varphi_j(x)\}_{j=1}^\infty$ the eigenvalues of problem (3.3) and eigenfunctions corresponding to $\{\mu_j\}_{j=1}^\infty$ and consider the integral equation [cf. Titchmarsh [5])

(3.4) $$\begin{aligned}\frac{1}{\mu_j+h}\varphi_j(x)&=p^{\frac{1}{2}}(x)\int_{R^n}H_{(\lambda,h)}(x,y)p^{\frac{1}{2}}(y)\varphi_j(y)dy\\&\quad+\frac{h}{\mu_j+h}p^{\frac{1}{2}}(x)\int_{R^n}H_{(\lambda,h)}(x,y)(p(x)-p(y))p^{-\frac{1}{2}}(y)\varphi_j(y)dy\\&\equiv a_j(x)+b_j(x)\qquad(j=1,2,\cdots)\end{aligned}$$

where $H_{(\lambda,h)}(x,y)=(2\pi)^{-n}\int_{R^n}\frac{e^{i(x-y)\cdot\xi}}{\lambda A(\xi)+1+hp(x)}d\xi$. By estimating $\int_{R^n}\sum_j a_j^2(x)dx$ and $\int_{R^n}\sum_j b_j^2(x)dx$, for any $\varepsilon>0$ we get

(3.5) $$\sum_j\frac{1}{(\mu_j+h)^2}=C_1\lambda^{-n/m}h^{n/l-2}+\varepsilon\lambda^{-n/m}h^{n/l-2}+\lambda^{-n/m}h^{n/l-2}C_1(\varepsilon)\lambda^\beta h^{-\alpha}$$

where $\alpha>\beta>0$ and $C_1(\varepsilon)$ is a constant independent of λ and h. From the Tauberian theorem of Hardy and Littlewood, we have for any $\varepsilon>0$

(3.6) $$N_\lambda(h)=C_2\lambda^{-n/m}h^{n/l}+\varepsilon\lambda^{-n/m}h^{n/l}\qquad\text{if } h\geq C_2(\varepsilon)\lambda^{\beta/\alpha}.$$

Since $\beta/\alpha<1$, we can put $h=\lambda$ in (3.6). Thus the theorem is proved when $p(x)$ is a positive smooth function.

In order to extend the result obtained above to the case that $p(x)=p_1(x)+p_2(x)$, we need some lemmas.

Lemma 1 (cf. Birman and Solomjak [2]). *Let $p_2(x)$ be a nonnegative function with compact support belonging to L_p, where $p=1$ if $m\geq n$ and $p>n/m$ if $m<n$. Let $M(h)$ be the number of eigenvalues less than h of the problem $Au=\lambda p_2u$. Then,*

$$M(h)=(2\pi)^{-n}\omega_0\int p_2(x)^{n/m}dx\,h^{n/m}+o(h^{n/m})$$

where $\omega_0=$meas $[\xi\,|\,A(\xi)\leq 1]$.

Lemma 2. *There is a constant $\varepsilon_0>0$ such that for any $\varepsilon<\varepsilon_0$, $(A+r)^{-1}p_2$ has at least one eigenvalue in $(\varepsilon/4,\varepsilon/3)$.*

Lemma 3. *Let $m(r,\varepsilon)$ be the number of eigenvalues greater than ε of operator $(A+r)^{-1}p_2$. Then,*

$$m(r,\varepsilon)\leq C_3(\varepsilon),$$

where $C_3(\varepsilon)$ is a constant independent of r.

Lemma 4. *For any $\varepsilon>0$, there is a constant $r(\varepsilon)$ such that for any $r<r(\varepsilon)$, $(A+r)^{-1}p_1$ has at least one eigenvalue in $(1-\varepsilon, 1)$.*

From Lemmas 1, 2, 3 and 4, we get for any $\varepsilon>0$,

$$n(r)\leq n\left(r, \frac{1}{1-\varepsilon}p_1\right)+C_4(\varepsilon) \tag{3.7}$$

where $n\left(r, \frac{1}{1-\varepsilon}p_1\right)$ is the number of eigenvalues less than $-r$ of the problem $Au-\frac{1}{1-\varepsilon}p_1u=\mu u$ and $C_4(\varepsilon)$ is a constant independent of r. From (3.7), we have

$$\overline{\lim_{r\to 0}}\, r^{n/l-n/m}n(r)\leq C_0(\varepsilon) \tag{3.8}$$

where $C_0(\varepsilon)=(2\pi)^{-n}\omega\frac{S}{n}a^{n/l}\left(\frac{1}{1-\varepsilon}\right)^{n/1}$.

Since $\varepsilon>0$ is arbitrary, we obtain

$$\overline{\lim_{r\to 0}}\, r^{n/l-n/m}n(r)\leq C_0(0) \tag{3.9}$$

It is not difficult to show that

$$\underline{\lim_{r\to 0}}\, r^{n/l-n/m}n(r)\geq C_0(0). \tag{3.10}$$

Thus we complete the proof of Theorem.

References

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