70. *Borel Structure in Topological $*$-algebras and Their Duals*

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1. Introduction. One of the useful tools for studying the structure of a locally compact group or Banach $*$-algebra A is the *dual space* $\hat{A}$ of all its equivalence classes of irreducible representations in Hilbert space. In this paper, we deal with the Borel structure of a dual space for a topological $*$-algebra. It will be shown that the dual space $\hat{\mathcal{D}}(G)$ of the topological $*$-algebra $\mathcal{D}(G)$, where G is a σ-compact Lie group, coinsides with the dual space $\hat{G}$ of the σ-compact Lie group G and that if in addition G satisfies some conditions the dual space $\hat{\mathcal{D}}(G)$ is an analytic Borel space.

From these results, we shall conclude that a connected semi-simple Lie group and a connected nilpotent Lie group are type 1.

For locally convex spaces and their related notions, see [6] and for Borel structures and their related notions, see [4]. The proofs are omitted, and the details will appear elsewhere. The author would like to express his thanks Prof. O. Takenouchi for his helpful comments.

2. Topological $*$ algebra. A *topological algebra* is an algebra and a topological vector space over the complex number field such that ring multiplication $\circ$ is jointly continuous. A topological algebra E with a mapping $*$ of E into itself is called a *topological $*$-algebra* if the following conditions are satisfied: (1) $(x^*)=x$, (2) $(x\circ y)^*=x^*\circ y^*$, (3) $(x+y)^*=x^*+y^*$, (4) $(\lambda x)^*=\bar{\lambda}x^*$ for every $x, y \in E$ and scalar λ. By a *representation*, we mean a mapping T of E into $\mathcal{L}(H, H)$, the set of all continuous linear mapping of a Hilbert space H into itself, which satisfies the following conditions: (1) $T(x+y)=T(x)+T(y)$, (2) $T(\lambda x)=\lambda T(x)$, (3) $T(x\circ y)=T(x)T(y)$, (4) $T(x^*)=T(x)^*$ for every $x, y \in E$ and scalar λ. A representation is said to be *cyclic* if there exists an element h_0 (which is called a *cyclic element* for T) in the Hilbert space H such that the set $\{T(x)h_0 \mid x \in E\}$ is dense in H. The continuity, the irreducibility and the equivalency are defined similarly to the case of the unitary representations of a topological group. A unitary representation U, of a topological group G in a Hilbert space H, is said to be *continuous* at g_0 if $U(g)h \to U(g_0)h$ as $g \to g_0$ in G for every $h \in H$.

In what follows by a representation, we shall mean a continuous representation.

Let G be a (non-compact) σ-compact (i.e., G can be represented as the union of countably many compact subsets G_n, $n \in N$, such that G_n is contained in the interior of G_{n+1} for all n) Lie group (for example, connected locally compact Lie group is a σ-compact Lie group). Let $\mathcal{D}(G_n)$ be the locally convex space of all complex-valued infinitely differentiable functions on G with support in G_n, is a Fréchet-space under the topology of uniform convergence in all derivaties. Let $\mathcal{D}(G)$ be the locally convex space of all complex-valued infinitely differentiable functions on G with compact supports, endowed with the strict inductive limit topology of the subspaces $\mathcal{D}(G_n)$: $\mathcal{D}(G) = \varinjlim \mathcal{D}(G_n)$. Then, the locally convex space $\mathcal{D}(G)$ may be topological $*$-algebra under the following ring multiplication and $*$-operation: for every $\phi, \psi \in \mathcal{D}(G)$ and $g \in G$,

$$(\phi \circ \psi)(g) = \phi * \psi(g) = \int \phi(g_1)\psi(g_1^{-1}g)d\mu(g_1)$$

where μ is a left invariant Haar measure on G, $\phi^*(g) = \chi^{-1}(g)\overline{\phi(g^{-1})}$ where $\chi(g)$ is a modular function on G. Moreover, $\mathcal{D}(G)$ is separable and a closed left ideal in $\mathcal{D}(G)$ is invariant under the left translations by the elements $g \in G$.

3. Relation between representations of the group G and those of the topological $*$-algebra $\mathcal{D}(G)$. Proposition 1. *Let G be a (non-compact) σ-compact Lie group with a left invariant Haar measure μ and $\mathcal{D}(G)$ be a topological $*$-algebra which is defined above. To each representation T of the topological $*$-algebra $\mathcal{D}(G)$ in a Hilbert space H there corresponds a unitary representation U of the σ-compact Lie group G in the Hilbert space H. Conversely, to each unitary representation U of the σ-compact Lie group G in a Hilbert space H there corresponds a reprentation T of the topological $*$-algebra $\mathcal{D}(G)$ in the Hilbert space H. These representations are connected by the formula*

$$T(\phi) = \int \phi(g)U(g)d\mu(g)$$

for every $\phi \in \mathcal{D}(G)$.

4. The Borel structure of the dual space of a separable topological $*$-algebra. Let E be a separable topological $*$-algebra. We denote by E^r the set of all equivalence classes of representations of E and by $\hat{E}$ the set of all equivalence classes of irreducible representations of E. We call $\hat{E}$ the *dual* of E. Let H^n be a Hilbert space of dimension n, and let H^∞ be a infinite dimensional separable Hilbert space. Let E^c denote the set of all representations T of E such that the representation space $H(T)$ is one of the spaces $H^1, H^2, \cdots, H^\infty$ without identification of equivalent representations. For each $T \in E^c$ let T^e denote the equivalence class to which it belongs. Then $T \to T^e$ is a map of E^c onto E^r. We denote by $E^{c,n}$ the subset of E^c consisting of all T with

$H(T)=H^n$. Let us consider the smallest Borel structure on E^c having the following two properties: (1) $E^{c,n}$ is a Borel subset of E^c for every n, (2) for each $n=1,2,\cdots$, each h and k in H^n and each $x\in E$, $T\to(T(x)h,k)$ is a Borel function on $E^{c,n}$. We shall consider the smallest Borel structure on E^r such that $T\to T^e$ is a Borel function. Finally we define on $\hat{E}$ the Borel structure which inherits as a subset of E^r. When we have the following.

Proposition 2. *If E is a topological $*$-algebra which is the strict inductive limit of complete, metrizable and separable locally convex spaces E_n: $E=\varinjlim E_n$, then E^c is a standard Borel space.*

5. The main theorem. Proposition 3. *Let G be a (non-compact) σ-compact Lie group such that the corresponding representation*

$$T(\phi)=\int\phi(g)U(g)d\mu(g)$$

of the topological $*$-algebra $\mathcal{D}(G)$ in a Hilbert space H, where U is a irreducible unitary representation of G in H, satisfies the following conditions: (1) *$T(\phi)$ is of the trace class for every $\phi\in\mathcal{D}(G)$, that is*

$$\text{Trace}\,(T(\phi))=\sum_{i=1}^{\infty}\int\phi(g)(U(g)h_i,h_i)d\mu(g)<\infty$$

where $\{h_i\}_{i=1}^{\infty}$ is an orthogonal basis in H, (2) Trace $(T(\phi))$ *is a continuous functional on $\mathcal{D}(G)$,* (3) *two irreducible representations T and L are equivalent representations if and only if* Trace $(T(\phi))=$ Trace $(L(\phi))$ *for every $\phi\in\mathcal{D}(G)$. Then the dual $\hat{\mathcal{D}}(G)$ of $\mathcal{D}(G)$ is an analytic Borel space.*

Now we consider the Borel structure in the duals of groups. We denote by G^r the set of all equivalence classes of unitary representations of G and by $\hat{G}$ the set of all equivalence classes of irreducible unitary representations of G. We call $\hat{G}$ the *dual* of G. Let us define Borel spaces $G^{c,n}$ and G^c analogously with $E^{c,n}$ and E^c. Further we define U^e as the equivalence class to which U belongs and define Borel structures on G^r and $\hat{G}$ by regarding G^r as a quotient space of G^c and $\hat{G}$ as a subspace of G^r. If $\hat{E}$ (resp. $\hat{G}$) is an analytic Borel space we say that it is *smooth* or that E (resp. G) has a *smooth dual*.

Lemma ([1]) 1. *Let G be separable locally compact group. Then G is type* 1 *if and only if G has a smooth dual.*

By Proposition 1, Proposition 3 and Lemma 1, we have the following.

Theorem. *Let G be a (non-compact) σ-compact Lie group G. If the topological $*$-algebra $\mathcal{D}(G)$ satisfies the three conditions* (1), (2) *and* (3) *of Proposition* 3, *then G is type* 1.

Lemma 2. *Let G be a connected semi-simple Lie group or a connected nilpotent Lie group. Let U be an irreducible unitary representation of G in a Hilbert space H and T the corresponding representation of the topological $*$-algebra $\mathcal{D}(G)$ (see, Proposition* 1). *Then T*

satisfies the three conditions (1), (2) *and* (3) *of Proposition* 3.

Lemma 2 is well known, see [2] for a connected semi-simple Lie group and see [3] and [5] for a connected nilpotent Lie group.

In virtue of Lemma 2 and Theorem, we have the following.

Corollary. *A connected semi-simple Lie group and a connected nilpotent Lie group are type* 1.

References

[1] J. Glimm: Type 1 C^*-algebras. Ann. Math., **73**, 572–612 (1961).

[2] Harish-Chandra: Representations of semi-simple Lie groups. Trans. Amer. Math. Soc., **76**, 234–253 (1954).

[3] A. A. Kirillov: Unitary representations of nilpotent Lie groups. Uspehi. Math. Nauk., **17**, 57–110 (1962).

[4] G. W. Mackey: Borel structure in groups and their duals. Trans. Amer. Math. Soc., **85**, 134–169 (1957).

[5] L. Pukanszky: Leçons sur les réprésentations des groupes. Paris (1967).

[6] H. H. Schaefer: Topological Vector Spaces. New York (1966).