69. *Closeness Spaces and Convergence Spaces*

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The purpose of this note is to show that every convergence structure ("Limitierung" of Fischer [2]) can be described by a family, called a *closeness*, of closure-like operations.

After stating several elementary properties of operations on the power set of a set, we shall introduce new notions "closeness" and "closeness space". Then some fundamental relations between closenesses and convergence structures will be established.

In what follows, the power set of a set X will be denoted by $\wp(X)$, and the value of a mapping $\alpha: \wp(X) \to \wp(X)$ at $A \in \wp(X)$ by A^α. The complement of $A \in \wp(X)$ in X will be written A^c. For each $x \in X$, $\dot{x}$ denotes the filter on X consisting of all $A \in \wp(X)$ with $x \in A$.

1. Throughout this section X denotes an arbitrary set. Let α be a mapping of $\wp(X)$ into itself. For each $x \in X$, we denote by $\Phi_\alpha(x)$ the set of all $A \in \wp(X)$ such that $x \notin A^{c\alpha}$. Evidently Φ_α is a mapping of X into $\wp\wp(X) = \wp(\wp(X))$.

The following four lemmas may be easily verified, and we omit the proofs.

Lemma 1. *Let α be a mapping of $\wp(X)$ into itself, and let $x \in X$. Then the following statements hold:*

(1) $\Phi_\alpha(x) \neq \emptyset$ *if and only if x does not belong to* $\cap\{A^\alpha | A \in \wp(X)\}$.

(2) $\emptyset \notin \Phi_\alpha(x)$ *if and only if* $x \in X^\alpha$.

Lemma 2. *Let α be a monotone mapping*) of $\wp(X)$ into itself. Then $x \in \{x\}^\alpha$ for every $x \in X$ if and only if $A \subset A^\alpha$ for every $A \in \wp(X)$.*

Lemma 3. *Let α be a monotone mapping of $\wp(X)$ into itself, and let $A \in \wp(X)$. Then $x \in A^\alpha$ if and only if $S \cap A \neq \emptyset$ for every $S \in \Phi_\alpha(x)$.*

Lemma 4. *Let α, β be two monotone mappings of $\wp(X)$ into itself. Then $\Phi_\alpha(x) \subset \Phi_\beta(x)$ for every $x \in X$ if and only if $A^\beta \subset A^\alpha$ for every $A \in \wp(X)$.*

Let Ψ be a mapping of X into $\wp\wp(X)$. For each $A \in \wp(X)$, we denote by $A^{\kappa(\Psi)}$ the set of all $x \in X$ for which we have $S \cap A \neq \emptyset$ for every $S \in \Psi(x)$. Obviously $\kappa(\Psi)$ is a monotone mapping of $\wp(X)$ into itself. Conversely, as an immediate consequence of Lemma 3, we have the following

*) A mapping α of $\wp(X)$ into itself is called *monotone* if $A \subset B$ implies $A^\alpha \subset B^\alpha$ for every $A, B \in \wp(X)$.

Lemma 5. *If α is a monotone mapping of $\wp(X)$ into itself, then $\alpha=\kappa(\Phi_\alpha)$.*

Now for each subset $\mathcal{A}$ of $\wp(X)$, we denote by $[\mathcal{A}]$ the set of all $S \in \wp(X)$ containing at least one member of $\mathcal{A}$.

Lemma 6. *Let Ψ be a mapping of X into $\wp\wp(X)$. Then*

$$\Phi_{\kappa(\Psi)}(x)=[\Psi(x)] \qquad \textit{for every } x \in X.$$

Proof. Clearly $A \in \Phi_{\kappa(\Psi)}(x)$ is equivalent to the fact that $S \cap A^c=\emptyset$ for some $S \in \Psi(x)$, and $S \cap A^c=\emptyset$ if and only if $S \subset A$.

By virtue of Lemma 5 and Lemma 6, we have at once the following

Corollary. *If α is a monotone mapping of $\wp(X)$ into itself, then*

$$[\Phi_\alpha(x)]=\Phi_\alpha(x) \qquad \textit{for every } x \in X.$$

Lemma 7. *Let α be a mapping of $\wp(X)$ into itself. If $(A \cup B)^\alpha = A^\alpha \cup B^\alpha$ for every $A, B \in \wp(X)$, then $\Phi_\alpha(x)$ is a filter on X for each $x \in X^\alpha \backslash \emptyset^\alpha$.*

Proof. Let $x \in X^\alpha \backslash \emptyset^\alpha$. Then by Lemma 1, the set $\Phi_\alpha(x)$ is non-empty and $\emptyset \notin \Phi_\alpha(x)$. On the other hand, the mapping α is monotone as can readily be seen. Hence according to the above Corollary we have $[\Phi_\alpha(x)]=\Phi_\alpha(x)$. Now if $A, B \in \Phi_\alpha(x)$, then since $x \notin A^{c\alpha}$ and $x \notin B^{c\alpha}$, we have

$$x \notin A^{c\alpha} \cup B^{c\alpha}=(A^c \cup B^c)^\alpha=(A \cap B)^{c\alpha},$$

which shows that $A \cap B \in \Phi_\alpha(x)$. This completes the proof.

Lemma 8. *For each mapping Ψ of X into $\wp\wp(X)$, the following statements hold:*

(1) *$\Psi(x) \neq \emptyset$ for every $x \in X$ if and only if $\emptyset^{\kappa(\Psi)}=\emptyset$.*

(2) *If $x \in X$, then $\emptyset \notin \Psi(x)$ if and only if $x \in X^{\kappa(\Psi)}$.*

Proof. To prove (1), suppose first $\emptyset^{\kappa(\Psi)} \neq 0$. Then there is an $x \in \emptyset^{\kappa(\Psi)}$. Hence if $\Psi(x)$ has a member S, then we have a contradiction $S \cap \emptyset \neq \emptyset$. Conversely if $\Psi(x)=\emptyset$ for some $x \in X$, then since $[\Psi(x)]=\emptyset$, we have, in view of Lemma 6 and (1) of Lemma 1,

$$x \in \cap \{A^{\kappa(\Psi)} | A \in \wp(X)\} \subset \emptyset^{\kappa(\Psi)},$$

and so $\emptyset^{\kappa(\Psi)} \neq \emptyset$. On the other hand, since $\emptyset \notin \Psi(x)$ if and only if $\emptyset \notin [\Psi(x)]$, the statement (2) follows immediately from Lemma 6 and (2) of Lemma 1.

A mapping α of $\wp(X)$ into itself is called a *semiclosure* on X if it satisfies the following conditions:

(1) $\emptyset^\alpha=\emptyset$ and $X^\alpha=X$.

(2) $(A \cup B)^\alpha=A^\alpha \cup B^\alpha$ for every $A, B \in \wp(X)$.

Lemma 7 yields obviously the following

Theorem 1. *If α is a semiclosure on X, then $\Phi_\alpha(x)$ is a filter on X for each $x \in X$.*

We have moreover the following

Theorem 2. *Let Ψ be a mapping of X into $\wp\wp(X)$. Then $\kappa(\Psi)$ is* *a semiclosure on X if and only if $\Psi(x)$ is a filter base on X for each $x \in X$.*

Proof. If $\kappa(\Psi)$ is a semiclosure on X, then by Lemma 7, $\Phi_{\kappa(\Psi)}(x)$ is a filter on X for each $x \in X$. But then since $\Phi_{\kappa(\Psi)}(x) = [\Psi(x)]$ by Lemma 6, $\Psi(x)$ is a filter base on X.

Conversely assume that $\Psi(x)$ is a filter base on X for each $x \in X$. Then $\emptyset^\alpha = \emptyset$ and $X^\alpha = X$ by Lemma 8. Let $x \in (A \cup B)^{\kappa(\Psi)}$. If $S \cap A \neq \emptyset$ for every $S \in \Psi(x)$, then $x \in A^{\kappa(\Psi)} \subset A^{\kappa(\Psi)} \cup B^{\kappa(\Psi)}$. If $S_0 \cap A = \emptyset$ for some $S_0 \in \Psi(x)$, then for each $S \in \Psi(x)$ the set $S \cap S_0$ contains some $S_1 \in \Psi(x)$, and hence we have

$$S \cap B = \emptyset \cup (S \cap B) = (S_0 \cap A) \cup (S \cap B)$$
$$\supset (S_1 \cap A) \cup (S_1 \cap B) = S_1 \cap (A \cup B) \neq \emptyset,$$

which implies that $x \in B^{\kappa(\Psi)} \subset A^{\kappa(\Psi)} \cup B^{\kappa(\Psi)}$. Thus $(A \cup B)^{\kappa(\Psi)} \subset A^{\kappa(\Psi)} \cup B^{\kappa(\Psi)}$. Now let x be in $A^{\kappa(\Psi)} \cup B^{\kappa(\Psi)}$; one can assume $x \in A^{\kappa(\Psi)}$. We have then

$$S \cap (A \cup B) = (S \cap A) \cup (S \cap B) \supset S \cap A \neq \emptyset$$

for every $S \in \Psi(x)$. It follows that $x \in (A \cup B)^{\kappa(\Psi)}$. Therefore $(A \cup B)^{\kappa(\Psi)} \supset A^{\kappa(\Psi)} \cup B^{\kappa(\Psi)}$. This completes the proof.

2. Let Γ be a set of semiclosures on a set X. The ordered pair (X, Γ) is called a *closeness space*, and Γ is called a *closeness* on X if the following conditions are satisfied:

(C1) For every $x \in X$, there exists an $\alpha \in \Gamma$ such that $x \in \{x\}^\alpha$.

(C2) For every $\alpha, \beta \in \Gamma$, there exists a $\gamma \in \Gamma$ such that $A^\alpha \cup A^\beta \subset A^\gamma$ for all $A \in \wp(X)$.

Let Γ, Γ' be two closenesses on a set X. We say that Γ' is *finer than* Γ(or Γ is *coarser than* Γ') if for every $x \in X$ and for every $\alpha \in \Gamma$, there exists a $\beta \in \Gamma'$ such that $\Phi_\beta(x) \subset \Phi_\alpha(x)$. Γ and Γ' are said to be *equivalent* or $\Gamma \equiv \Gamma'$ if Γ is finer than Γ' and if Γ' is finer than Γ. It is easy to see that $\equiv$ is an equivalence relation on the set of all closenesses on X.

Theorem 3. *Let X be a set. For each closeness Γ on X, there exists a unique convergence structure τ on X such that, for every $x \in X$, $\Psi \in \tau(x)$ if and only if $\Phi_\alpha(x) \subset \Psi$ for some $\alpha \in \Gamma$.*

Proof. It clearly suffices to show that the mapping τ of X into the power set of the set $\boldsymbol{F}(X)$ of all filters on X defined by

$$\tau(x) = \{\Psi \in \boldsymbol{F}(X) \mid \Phi_\alpha(x) \subset \Psi \text{ for some } \alpha \in \Gamma\} \qquad \text{for every } x \in X,$$

is a convergence structure on X. Theorem 1 shows that the mapping τ is well-defined. Let $x \in X$ and $\Phi, \Psi \in \tau(x)$. Then we have $\Phi_\alpha(x) \subset \Phi$ and $\Phi_\beta(x) \subset \Psi$ for some $\alpha, \beta \in \Gamma$. Hence the condition (C2) ensures the existence of a $\gamma \in \Gamma$ such that $A^\alpha \cup A^\beta \subset A^\gamma$ for all $A \in \wp(X)$. Now if $A \in \Phi_\gamma(x)$, then since $x \notin A^{c\gamma}$, we have $x \notin A^{c\alpha}$ and $x \notin A^{c\beta}$, which imply

$$A \in \Phi_\alpha(x) \cap \Phi_\beta(x) \subset \Phi \cap \Psi.$$

Consequently we have $\Phi_\gamma(x) \subset \Phi \cap \Psi$, and hence $\Phi \cap \Psi \in \tau(x)$. It remains to prove that $\dot{x} \in \tau(x)$ for each $x \in X$. Let x be in X. Then by (C1) one can find an $\alpha \in \Gamma$ such that $x \in \{x\}^{\alpha}$. If A is a member of $\Phi_{\alpha}(x)$, then since $x \notin A^{c\alpha}$, the set A^{c} cannot contain $\{x\}$, and so $x \in A$. Thus we have $\Phi_{\alpha}(x) \subset \dot{x}$ as desired.

The convergence structure whose existence is ensured by Theorem 3 is called the *convergence structure associated with* Γ and is denoted by τ_{Γ}, that is

$$\tau_{\Gamma}(x) = \{\Psi \in \boldsymbol{F}(X) \mid \Phi_{\alpha}(x) \subset \Psi \qquad \text{for some } \alpha \in \Gamma\}$$

for every $x \in X$, where $\boldsymbol{F}(X)$ denotes the set of all filters on X.

It is easy to verify the following theorem, and the proof is therefore omitted.

Theorem 4. *Let Γ, Γ' be two closenesses on a set X. Then Γ is finer than Γ' if and only if τ_{Γ} is finer than $\tau_{\Gamma'}$.*

Thus we have the following

Corollary. *Two closenesses Γ, Γ' on a set X are equivalent if and only if $\tau_{\Gamma} = \tau_{\Gamma'}$.*

We shall now prove the following

Theorem 5. *For each convergence structure τ on X, there exists a closeness Γ on X such that $\tau = \tau_{\Gamma}$. The closeness Γ can be chosen to satisfy moreover the condition*

(C1′) *There exists an $\alpha \in \Gamma$ such that $A \subset A^{\alpha}$ for every $A \in \wp(X)$.*

Proof. Let Γ denotes the set of all $\kappa(\Psi)$ where Ψ runs through the set $\prod\{\tau(x) \mid x \in X\}$. By Theorem 2, each element of Γ is a semiclosure on X. We shall show that Γ satisfies the condition (C1′) which implies (C1). Since $\dot{x} \in \tau(x)$ for each $x \in X$, there is a $\Psi \in \prod\{\tau(x) \mid x \in X\}$ such that $\Psi(x) = \dot{x}$ for every $x \in X$; by Lemma 2, it is sufficient to prove that $x \in \{x\}^{\kappa(\Psi)}$ for every $x \in X$. Let $x \in X$; then for each $S \in \Psi(x)$, we have $S \cap \{x\} \neq \emptyset$, and consequently $x \in \{x\}^{\kappa(\Psi)}$. In order to verify (C2), let $\alpha, \beta \in \Gamma$. Then $\alpha = \kappa(\Psi_1)$ and $\beta = \kappa(\Psi_2)$ for some $\Psi_1, \Psi_2 \in \prod\{\tau(x) \mid x \in X\}$, and hence we can find a $\Psi_0 \in \prod\{\tau(x) \mid x \in X\}$ such that $\Psi_0(x) = \Psi_1(x) \cap \Psi_2(x)$ for all $x \in X$. Let us denote by γ the semiclosure $\kappa(\Psi_0) \in \Gamma$, and let $A \in \wp(X)$. If $x \in A^{\alpha}$, then since $\Psi_0(x) \subset \Psi_1(x)$, we have $S \cap A \neq \emptyset$ for every $S \in \Psi_0(x)$, which shows that $x \in A^{\gamma}$. It follows that $A^{\alpha} \subset A^{\gamma}$. Thus we have $A^{\alpha} \cup A^{\beta} \subset A^{\gamma}$. It remains only to prove that $\tau = \tau_{\Gamma}$. Let x be in X. For each $\mathscr{F} \in \tau(x)$, one can find a $\Psi \in \prod\{\tau(x) \mid x \in X\}$ for which we have $\Psi(x) = \mathscr{F}$; then since $\Phi_{\kappa(\Psi)}(x) = [\Psi(x)] = [\mathscr{F}] = \mathscr{F}$ by Lemma 6, we have $\mathscr{F} \in \tau_{\Gamma}(x)$. Consequently $\tau(x) \subset \tau_{\Gamma}(x)$. Conversely for each $\mathscr{F} \in \tau_{\Gamma}(x)$, there is a $\Psi \in \prod\{\tau(x) \mid x \in X\}$ such that $\Phi_{\kappa(\Psi)}(x) \subset \mathscr{F}$; and hence by Lemma 6 again, we have $\Phi_{\kappa(\Psi)}(x) = [\Psi(x)] = \Psi(x) \in \tau(x)$ which implies $\mathscr{F} \in \tau(x)$. Therefore we have $\tau_{\Gamma}(x) \subset \tau(x)$. Thus $\tau(x) = \tau_{\Gamma}(x)$ for every $x \in X$.

As an immediate consequence of Theorem 5 and Corollary of Theorem 4, we have the following

Corollary. *For each closeness Γ on a set X, there exists a closeness Γ' on X satisfying the conditions* (C1′) *and $\Gamma \equiv \Gamma'$.*

Let X be a set and let α be a mapping of $\wp(X)$ into itself. Then by Lemma 2, if $\{\alpha\}$ is a closeness on X then $A \subset A^\alpha$ for every $A \in \wp(X)$. Consequently $\{\alpha\}$ is a closeness on X if and only if the following conditions are satisfied:

(P1) $\emptyset^\alpha = \emptyset$.

(P2) $A \subset A^\alpha$ for every $A \in \wp(X)$.

(P3) $(A \cup B)^\alpha = A^\alpha \cup B^\alpha$ for every $A, B \in \wp(X)$.

In other words, $\{\alpha\}$ is a closeness on X if and only if α is a structure of "pré-adhérence" of Choquet [1]. An operator α satisfying the conditions (P1)–(P3) is called a *closure topology* by Koutník [3]. On the other hand, Rehermann [4] has introduced the notions of "liaison" and "liaison space": a subset λ of $X \times (\wp(X) \setminus \{\emptyset\})$ is called a *liaison* and the pair (X, λ) a *liaison space* if

(L1) $x\lambda\{x\}$ for every $x \in X$, and

(L2) $x\lambda(A \cup B)$ if and only if $x\lambda A$ or $x\lambda B$, for every $x \in X$ and for every $A, B \in \wp(X)$,

where $x\lambda A$ means $(x, A) \in \lambda$. In a liaison space (X, λ), Rehermann defines the *capusle* $A^{\alpha(\lambda)}$ of each $A \in \wp(X)$ by

$$A^{\alpha(\lambda)} = \{x \in X \mid x\lambda A\}.$$

As is shown in [4], the mapping $\alpha(\lambda)$ of $\wp(X)$ into itself satisfies the conditions (P1)–(P3), and hence $\{\alpha(\lambda)\}$ is a closeness on X. Conversely if $\{\alpha\}$ is a closeness on X, then as can be easily seen, we have $\alpha = \alpha(\lambda)$ for the liaison $\lambda = \{(x, A) \in X \times (\wp(X) \setminus \{\emptyset\}) \mid x \in A^\alpha\}$ on X. Thus a liaison and a closeness consisting of a single element define the same kind of structures. Moreover the structures of "pré-adhérence" of Choquet coincide with the principal convergence structures ("Hauptideal-Limitierung" of Fischer. See [2]). This leads us to the following

Theorem 6. *A closeness Γ on a set X is equivalent to a closeness on X which is a singleton if and only if τ_Γ is a principal convergence structure on X.*

Proof. It will be enough to prove the "if part". Assume that τ_Γ is a principal convergence structure on X. Then for each $x \in X$, there exists a unique filter $\Psi(x)$ on X such that $\tau_\Gamma(x)$ is the set of all filters on X finer than $\Psi(x)$. In order to prove that $\Gamma' = \{\kappa(\Psi)\}$ is a closeness on X, it clearly suffices to show that $x \in \{x\}^{\kappa(\Psi)}$ for every $x \in X$. To this end, let $x \in X$. Then since $\Psi(x) \subset \dot{x}$, we have $S \cap \{x\} \neq \emptyset$ for every $S \in \Psi(x)$, and hence we have $x \in \{x\}^{\kappa(\Psi)}$. Therefore Γ' is a closeness on X. Now by Lemma 5, we have

$$\begin{aligned}\tau_{\Gamma'}(x) &= \{\mathcal{F} \in \boldsymbol{F}(X) \mid \Phi_{\kappa(\Psi)}(x) \subset \mathcal{F}\} = \{\mathcal{F} \in \boldsymbol{F}(X) \mid [\Psi(x)] \subset \mathcal{F}\} \\ &= \{\mathcal{F} \in \boldsymbol{F}(X) \mid \Psi(x) \subset \mathcal{F}\} = \tau_\Gamma(x)\end{aligned}$$

for each $x \in X$, where $\boldsymbol{F}(X)$ denotes the set of all filters on X. Hence it follows from Corollary of Theorem 5 that Γ' and Γ are equivalent.

References

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