64. Wave Equation with Wentzell's Boundary Condition and a Related Semigroup on the Boundary. II

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1. In part I of this paper [1], we defined a closure $\overline{A}_L$ of A with respect to Wentzell's *boundary condition*

$$Lu(x)=0, \qquad x \in \partial D,$$

and solved the *wave equation*

$$(1) \qquad \frac{\partial^2}{\partial t^2}u=\overline{A}_L u, \quad u(t,\cdot)\to f, \quad \frac{\partial}{\partial t}u(t,\cdot)\to g, \qquad \text{as } t\to 0,$$

by solving the equations of type

$$(2) \qquad \alpha u-\overline{A}_L u=v, \qquad \text{for } v\in \mathcal{H},$$

and using the scheme in 2 of [1].

Here, we consider L as an operator which maps a function u on $\overline{D}$ to a function Lu on ∂D, and define a closure $\overline{L}_A$ of L with respect to the *domain condition*

$$(3) \qquad Au(x)=0, \qquad x\in D,$$

just as we defined $\overline{A}_L$. Since each function in $\mathcal{D}(\overline{L}_A)$ can be proved to satisfy (3), it is written as $H\varphi(x)=\int_{\partial D} H(x,dy)\varphi(y)$ by the boundary value φ and the harmonic measure $H(x,\cdot)$ with respect to the domain D and point x.[1] Thus, we define $\overline{LH}$ by $\overline{LH}\varphi=\overline{L}_A H\varphi$ on $\{\varphi\in\mathcal{H}_\partial \mid H\varphi \in D(\overline{L}_A)\}$, where $\mathcal{H}_\partial$ is the Hilbert space of all measurable functions on ∂D such that $\|\varphi\|_\partial=\langle\varphi,\varphi\rangle^{\frac{1}{2}}<\infty$. Then, we can solve

$$(4) \qquad \frac{\partial^2}{\partial t^2}\varphi=\overline{LH}\varphi, \quad \varphi(t,\cdot)\to\psi, \quad \frac{\partial}{\partial t}\varphi(t,\cdot)\to\eta, \qquad \text{as } t\to 0,$$

by using the scheme in 2 of [1] and solving the equations of type

$$(5) \qquad \lambda[u]_\partial-\overline{L}_A u=\varphi, \qquad \text{for } \varphi\in\mathcal{H}_\partial,$$

where $[u]_\partial$ is the restriction of u to the boundary ∂D.

It is expected that the mapping L and the equation (4) have some intuitive meanings, closely related with (1). Some comments on this point will be added in comparison with equation

$$(6) \qquad \frac{\partial}{\partial t}\varphi=\overline{LH}\varphi, \quad \varphi(t,\cdot)\to\psi, \qquad \text{as } t\to 0,$$

1) The harmonic measure corresponds to $A=\Delta$. For a general A, a measure with similar properties exists, and it is sometimes called the *hitting measure*. In fact, this is the probability distribution of the first hit to the boundary of the diffusion particle corresponding to A and started at point x.

which corresponds to the diffusion equation

$$(7)\qquad \frac{\partial}{\partial t}u=\overline{A}_L u,\quad u(t,\cdot)\to f,\qquad \text{as } t\to 0.$$

2. For f, g in $\mathcal{H}_0$ and $\lambda\geqslant 0$, we define

$$B^\lambda(f,g)=\lambda\langle f,g\rangle+D(f,g)+a\cdot D\langle f,g\rangle+\nu(f,g).$$

By the known estimates

$$(8)\quad c\,\|f\|_\partial^2\leqslant\|f\|^2+D(f,f),\quad c\,\|f\|^2\leqslant\|f\|_\partial^2+D(f,f),\qquad \text{for } f\in\mathcal{H}_0,$$

$B^\lambda(f,g)$ is equivalent with $B_\alpha(f,g)$ for positive λ and α,[2] as in

Proposition 1. $\langle\lambda f-Lf,g\rangle-(Af,g)_s=B^\lambda(f,g)$, $f,g\in\mathcal{H}_0$, $\lambda\geqslant 0$. *$B^\lambda(f,g)$ can be extended uniquely to a bilinear functional on $\mathcal{K}$. The extension, under the same notation, satisfies*

$$B^\lambda(f,g)\leqslant c_\lambda\,\|f\|_l\,\|g\|_l\,,\quad \|f\|_l^2\leqslant c_\lambda B^\lambda(f,f),\qquad \textit{for } f,g\in\mathcal{K} \textit{ and } \lambda>0.$$

Proposition 2. *If $\{f_n, n=1,2,\cdots\}$ in $\mathcal{H}_0$ and $\varphi\in\mathcal{H}_\partial$ satisfy $\lim_{n\to\infty}\|f_n\|_l=0$ and $\lim_{n\to\infty}\{(Af_n,h)_s+\langle Lf_n-\varphi,h\rangle\}=0$ for each $h\in\mathcal{H}_0$, then $\varphi=0$.*

Definition 1. If, for $f\in\mathcal{K}$, there are a sequence $\{f_n, n=1,2,\cdots\}$ in $\mathcal{H}_0$ and φ in $\mathcal{H}_\partial$ such that $\lim_{n\to\infty}\|f_n-f\|_l=0$, and

$$(9)\qquad \lim_{n\to\infty}\{(Af_n,h)_s+\langle Lf_n-\varphi,h\rangle\}=0,\qquad \text{for each } h\in\mathcal{H}_0,$$

then we define $\overline{A}_L f=\varphi$, and denote the set of all such f by $\mathcal{D}(\overline{L}_A)$.

Proposition 3. *f in $\mathcal{K}$ belongs to $\mathcal{D}(\overline{L}_A)$, if and only if there is a φ in $\mathcal{H}_\partial$ such that*

$$B^\lambda(f,h)=\langle\varphi,h\rangle,\qquad \textit{for } h\in\mathcal{H}_0.$$

In this case, φ satisfies

$$\lambda[f]_\partial-\overline{L}_A f=\varphi.$$

Proposition 4. *For each $\varphi\in\mathcal{H}_\partial$ and $\lambda>0$,* (5) *has a unique solution f such that*

$$\|f\|_l\leqslant c'_\lambda\,\|\varphi\|_\partial,\quad B^\lambda(f,g)=\langle\varphi,g\rangle\qquad \textit{for } g\in\mathcal{K}.$$

Hence, $\lambda-\overline{L}_A$ maps $\mathcal{D}(\overline{L}_A)$ onto $\mathcal{H}_\partial$ in one to one way, and $(\lambda-\overline{L}_A)^{-1}$ is linear and bounded.

The proof is similar to the case of (2), considering $F(f)=\langle\varphi,f\rangle$ for $f\in\mathcal{K}$ in the place of $F(f)=(v,f)_s$ for Proposition 5 in [1].

Proposition 5. *Each f in $\mathcal{D}(\overline{L}_A)$ satisfies* (3).

In fact, let $\{f_n\}$ be a sequence in $\mathcal{H}_0$ such that (9) holds, and let h be in $\mathcal{H}_0$ and vanish near ∂D. Then, by Green-Stokes formula, we have

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2) For a more general description, it is natural to define

$$B_\alpha^\lambda(f,g)=\alpha(f,g)+\lambda\langle f,g\rangle+D(f,g)+aD\langle f,g\rangle+\nu(f,g),$$

instead of introducing $B_\alpha(f,g)$ and $B^\lambda(f,g)$ separately. Then, a duality between $\overline{A_L}$ and $\overline{L_A}$ extends to $\overline{A_{L-\lambda}}$ and $\overline{L_{A-\alpha}}$, and a relation between G_α and $\overline{LH_\alpha}$ can be discussed as in [2]. But, this is not necessary for our present purpose, and we omit it.

$$0=\lim_{n\to\infty}\{(Af_n, h)_s+\langle Lf_n-\bar{L}_A f, h\rangle\}$$
$$=\lim_{n\to\infty}(Af_n, h)_s=\lim_{n\to\infty}(f_n, Ah)=(f, Ah),$$

which implies the above assertion.

3. Semigroup on the boundary. Since $\bar{D}$ is compact and ∂D is smooth, there is a unique solution of

$$Au(x)=0, \quad x\in D, \quad u(x)=\varphi(x), \quad x\in\partial D, \qquad \text{for } \varphi\in C(\partial D).$$

The solution is written as $u(x)=H\varphi(x)=\int_{\partial D}H(x, dy)\varphi(y)$ by a measure $H(x, \cdot)$ on ∂D with total mass 1. By the known estimate

(10) $$\|H\varphi\|\leqslant c'\|\varphi\|_\partial, \qquad \text{for } \varphi\in C(\partial D),$$

H can be extended uniquely to a bounded linear mapping from $\mathcal{H}_\partial$ to $\mathcal{H}$. The extension, under the same notation, satisfies

$$\|H\varphi\|_s\leqslant c''\|\varphi\|_\partial, \qquad \varphi\in\mathcal{H}_\partial.$$

We write $\mathcal{H}_{0,\partial}$ for the set of all $[f]_\partial$ of f in $\mathcal{H}_0$, that is, $\mathcal{H}_{0,\partial}=\{[f]_\partial \mid f\in\mathcal{H}_0\}$. We define, for φ, ψ in $\mathcal{H}_{0,\partial}$,

$$B_\lambda\langle\varphi, \psi\rangle=B^\lambda(H\varphi, H\psi)$$
$$\langle\varphi, \psi\rangle_l=B_1\langle\varphi, \psi\rangle, \qquad \|\varphi\|_{l,\partial}=\langle\varphi, \varphi\rangle_l^{\frac{1}{2}}.$$

Let $\mathcal{K}_\partial$ be the completion of $\mathcal{H}_{0,\partial}$ with respect to $\|\ \|_{l,\partial}$. $B_\lambda(\cdot, \cdot)$, $\langle\cdot, \cdot\rangle_l$ and $\|\ \|_{l,\partial}$, are extended on $\mathcal{K}_\partial$. $\mathcal{K}_\partial$ is imbedded in $\mathcal{H}_\partial$ as a dense subset.

Proposition 6. *For f in $\mathcal{K}$, $[f]_\partial$ belongs to $\mathcal{K}_\partial$.*

In fact, there is a sequence $\{f_n\}$ in $\mathcal{H}_0$ such that $\|f_n-f\|_l\to 0$. But, an arbitrary h in $\mathcal{H}_0$ is written as $h=H[h]_\partial+g$, where $g=h-H[h]_\partial$ is smooth and vanishes on ∂D. Since $H[h]_\partial$ satisfies (3), $D(H[h]_\partial, g)=0$. Thus, by a simple computation using (8),

$$\|h\|_l^2=\|H[h]_\partial\|_l^2+D(g, g)\geqslant\|H[h]_\partial\|_l^2\geqslant\frac{1}{2}\min(c, 1)\|[h]_\partial\|_{l,\partial}^2.$$

This, applied for $h=f_m$ or f_n, implies $\lim_{m\to\infty, n\to\infty}\|(f_m)_\partial-(f_n)_{l,\partial}\|\leqslant c'\cdot\lim_{m\to\infty,n\to\infty}\|f_n-f_m\|_l=0$. Thus, $[f_n]_\partial$ converges to a limit in $\mathcal{K}_\partial$, which coincides with $[f]_\partial$ by $c\|[f]_\partial-[f_n]_\partial\|_\partial\leqslant\|f_n-f\|_l\to 0$.

Definition 2. Let $\mathcal{D}(\overline{LH})$ be the set of all φ in $\mathcal{H}_\partial$ such that $H\varphi\in\mathcal{D}(\bar{L}_A)$. We define $\overline{LH}\varphi=\bar{L}_A H\varphi$, for $\varphi\in\mathcal{D}(\overline{LH})$.

$\mathcal{D}(\overline{LH})$ is contained in $\mathcal{K}_\partial$. In fact, for each φ in $\mathcal{D}(\overline{LH})$, $H\varphi$ is in $\mathcal{D}(\bar{L}_A)$, and hence in $\mathcal{K}$. Thus, by Proposition 6, $[H\varphi]_\partial=\varphi$ is in $\mathcal{K}_\partial$.

Here, we rewrite Proposition 4 as in

Lemma 1. (11) *$\langle\lambda\varphi-\overline{LH}\varphi, \psi\rangle=B_\lambda\langle\varphi, \psi\rangle$, for $\varphi\in\mathcal{D}(\overline{LH})$, $\psi\in\mathcal{K}_\partial$. There is a unique solution of*

$$\lambda\varphi-\overline{LH}\varphi=\psi, \quad \textit{for each } \psi\in\mathcal{H}_\partial \textit{ and } \lambda>0.$$

Thus, $K_\lambda=(\lambda-\overline{LH})^{-1}$ is defined on $\mathcal{H}_\partial$, and maps $\mathcal{H}_\partial$ onto $\mathcal{D}(\overline{LH})$ in one to one way, satisfying

$$\|K_\lambda\psi\|_{l,\partial}\leqslant c''\|\psi\|_\partial, \quad B_\lambda\langle K_\lambda\psi, \eta\rangle=\langle\psi, \eta\rangle \qquad \text{for each } \eta\in\mathcal{K}_\partial.$$

Proposition 7. *For* $\varphi=K_\lambda\psi$, *we have*

$$\lambda\|\varphi\|_\partial^2+(\|\varphi\|_{l,\partial}^2-\|\varphi\|_\partial^2)=\langle\varphi,\psi\rangle \qquad \text{for } \psi\in\mathcal{H}_\partial.$$

$$\|\psi-\lambda\varphi\|_\partial^2+\lambda(\|\varphi\|_{l,\partial}^2-\|\varphi\|_\partial^2)=\langle\varphi,\psi\rangle_l-\langle\varphi,\psi\rangle, \qquad \text{for } \psi\in\mathcal{K}_\partial.$$

$$\lambda\|\psi-\lambda\varphi\|_l^2+(\|\psi-\lambda\varphi\|_{l,\partial}^2-\|\psi-\lambda\varphi\|_\partial^2)=\langle\lambda\varphi-\psi,\overline{LH}\psi\rangle, \qquad \text{for } \psi\in\mathcal{D}(\overline{LH}).$$

These are proved by using (11). Combining these equalities, we have

Lemma 2. $\lambda\|K_\lambda\varphi\|_\partial\leqslant\|\varphi\|_\partial$, $\lim_{\lambda\to\infty}\|\lambda K_\lambda\varphi-\varphi\|_\partial=0$, *for* $\varphi\in\mathcal{H}_\partial$.

$$\lambda\|K_\lambda\varphi\|_{l,\partial}\leqslant\|\varphi\|_{l,\partial},\quad \lim_{\lambda\to\infty}\|\lambda K_\lambda\varphi-\varphi\|_{l,\partial}=0, \qquad \text{for } \varphi\in\mathcal{K}_\partial.$$

Theorem 1. *$\overline{LH}$ is the generator of a semigroup $\{\tilde{T}_t, t\geqslant 0\}$ on $\mathcal{H}_\partial$, which satisfies $(A,2)$ and $(A,3)$ in 2 of* (1). *$(A,1)$, $(A,2)$ and $(A,5)$ are satisfied for $\mathcal{A}=\overline{LH}$ and for $\mathcal{H}_\partial$ and $\mathcal{K}_\partial$, replaced in the place of $\mathcal{H}$ and $\mathcal{K}$. Hence, there is a group of bounded linear operators $\{\tilde{U}_t, -\infty<t<\infty\}$ on the space $\tilde{\boldsymbol{B}}=\begin{pmatrix}\mathcal{K}_\partial\\ \mathcal{H}_\partial\end{pmatrix}$ with norm $\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|=(\|\varphi\|_{l,\partial}^2+\|\psi\|_\partial^2)^{\frac{1}{2}}$. $\{\tilde{U}_t\}$ satisfies $\|\tilde{U}_t\|\leqslant e^{b''t}$ and has generator $\tilde{\boldsymbol{G}}$:*

$$\tilde{\boldsymbol{G}}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\begin{pmatrix}\psi\\ \overline{LH}\varphi\end{pmatrix}, \quad \text{for} \quad \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\in\mathcal{D}(\tilde{\boldsymbol{G}})=\begin{pmatrix}\mathcal{D}(\overline{LH})\\ \mathcal{K}_\partial\end{pmatrix}.$$

Hence, the unique solution of (4) *is given by $\tilde{U}_t\begin{pmatrix}\psi\\ \eta\end{pmatrix}$ for $\psi\in\mathcal{D}(\overline{LH})$ and $\eta\in\mathcal{K}_\partial$.*

The proof is similar to that of Theorem 2 in (1).

4. In the case of the diffusion equation, the terms in $Lu(x)$ have the intuitive meanings: $\sum\alpha_{ij}(x)\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}(x)+\sum\beta_i(x)\frac{\partial u}{\partial\xi_i}(x)$ corresponds to the diffusing along the boundary ∂D, $\gamma(x)u(x)$ to the vanishing of the particle at ∂D, $\delta(x)Au(x)$ to the *sticky barrier* where the particle spends time comparably long with the stay in the domain D. $\mu(x)\frac{\partial u}{\partial n}(x)$ corresponds to the reflection at ∂D, and the last term to the jump ∂D according to the measure $\nu(x,\cdot)$.

For a smooth function φ on ∂D, $\overline{LH}\varphi$ can be represented as

$$\overline{LH}\varphi(x)=\sum\tilde{\alpha}_{ij}(x)\frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j}(x)+\sum\tilde{\beta}_i(x)\frac{\partial\varphi}{\partial\xi_i}(x)+\tilde{\gamma}(x)\varphi(x)$$
$$+\int_{\partial D}\left(\varphi(y)-\varphi(y)-\sum\frac{\partial\varphi}{\partial\xi_i}(x)\xi_i(x,y)\right)\tilde{\nu}(x,dy)$$

The semigroup with generator $\overline{LH}$, in the set up of [2], corresponds to a Markov process on the boundary, which is the trace on ∂D of the diffusion determined by [7], described by a time scale called the *local time on the boundary*. This was conjectured and proved in a special case in [2], and extended by K. Sato [3] and then by P. Priouret [4] for a wide class of Wentzell's boundary conditions.[3]

For the wave equation, a kind of *duality* in appearence between (2) and (5) seems to suggest, as in the case of diffusion, that the solution of (4), or the group of operators with generator $\tilde{G}\begin{pmatrix}\varphi \\ \psi\end{pmatrix}=\begin{pmatrix}\psi \\ LH\varphi\end{pmatrix}$, describes the wave propagation restricted on the boundary, depending on a time scale for the boundary ∂D. Here, it is expected that the boundary has a mass distributed according to the measure $\delta(x)dx$,[4] and the wave propagates through ∂D partly by the vibration of the boundary itself determined by the term

$$\sum \alpha_{ij}(x)\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}(x)+\sum \beta_i(x)\frac{\partial u}{\partial\xi_i}(x),$$

just as the wave propagation in D is determined by

$$Au(x)=\sum a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}(x)+\sum b_i(x)\frac{\partial u}{\partial x_i}(x).$$

The classical terms $\gamma(x)u(x)$ and $\mu(x)\frac{\partial u}{\partial n}(x)$ correspond to the energy loss and the reflection at ∂D, respectively. By the last term of $Lu(x)$, a wave arrived at point x on ∂D instantly gives effect on the support of the measure $\nu(x,\cdot)$.

In the case of diffusion equation, the above explanations are justified rigorously on the basis of path spaces and the related mathematical tools. On the other hand, it seems that a parallel justification for the wave equation is not possible at present. Some mathematical method for a more detailed description of the wave propagation is desired.

References

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3) Markov process on the boundary can be obtained by a purely probabilistic approach as in K. Sato [5] and M. Motoo [6], by defining the local time on the boundary first.

4) This interpretation of the term $\delta(x)Au(x)$ was given by Feller [7] in one dimension. In view of the definition of $\mathcal{H}$, this naturally extends to the general case.

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