61. On Micro-Analyticity of the Elementary Solutions of Hyperbolic Differential Equations with Real Analytic Coefficients

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(Comm. by Kôsaku YOSIDA, M. J. A., April 18, 1974)

In this note we state a theorem on (micro-) analyticity of the elementary solutions of hyperbolic differential equations with (not necessarily constant) multiple characteristics. Our result is a generalization of those of Kawai [1], Hörmander [2] and Andersson [3] which deal with operators with simple characteristics. (See Atiyah-Bott-Gårding [4] for operators with constant coefficients.)

If an m-th order differential operator $P(t, x, D_t, D_x)$ is hyperbolic with respect to the direction $(1, \cdots, 0)$, there exists a unique elementary solution of the Cauchy problem, that is, m-tuple of hyperfunctions $E_j(t, x)$ $(j=1, \cdots, m)$ such that

$$P(t, x, D_t, D_x)E_j(t, x)=0,$$
$$D_t^{i-1}E_j(0, x)=\delta_{ij}\delta(x) \qquad (i, j=1, \cdots, m).$$

(See Kawai [5] and Bony-Schapira [6].) Our problem is to decide the singular spectrum of $E_j(t, x)$.

Recently Kashiwara-Kawai [7] defined micro-hyperbolicity and constructed good elementary solutions for micro-hyperbolic operators. The essential key to our theorem is their deep analysis in micro-local sense. Remark that our lemma is valid for pseudo-differential operators.

Here we treat only the simplest case. More complete results and proofs will be published elsewhere.

First we set up a class of operators which can be easily handled. Let $P(x, D_x)$ be a pseudo-differential operator defined in a neighborhood of $x_0^*=(x_0, \xi_0) \in P^*X$. Let $\sigma(P)(x, \xi)=p_1^{s_1}(x, \xi)\cdots p_r^{s_r}(x, \xi)$ be an irreducible decomposition at x_0^*. We call $P(x, D_x)$ reductive if each $p_j(x, \xi)$ is simple characteristic, that is, $d_{(x,\xi)}p_j(x, \xi)$ is not parallel to $\sum_i \xi_i dx_i$. In this case we can define r-bicharacteristic strips through x_0^*. A hyperbolic differential operator is called reductive if it is reductive at each point on its real characteristic variety.

Examples.

$$D_t^2-t^2(D_x^2+D_y^2)$$
$$(D_t^2-a(t, x, y)D_x^2-b(t, x, y)D_y^2)(D_t^2-c(t, x, y)D_x^2-d(t, x, y)D_y^2)$$

where a, b, c and d is positive for real (t, x, y).

Now let us define characteristic conoid of a reductive hyperbolic differential operator. Let $P(t, x, D_t, D_x)$ be a reductive hyperbolic differential operator with respect to the direction $(1, \cdots, 0)$ defined in a neighborhood of the origin. Let $V_R=\{(t, x, \tau, \xi) \in \sqrt{-1}S^*M / \sigma(P)(t, x, \tau, \xi)=0\}$ and $D=\{(t, x, \tau, \xi) \in V_R /$the number of bicharacteristic strips through $(t, x, \tau, \xi) \geqq 2\}$. We assume the following: at each point $(t_0, x_0, \tau_0, \xi_0) \in D$, the number of bicharacteristic strips is two and if $\sigma(P)(t, x, \tau, \xi)=p_1^{s_1}(t, x, \tau, \xi)p_2^{s_2}(t,x,\tau,\xi)$ is an irreducible decomposition, $\{p_1, p_2\}(t_0, x_0, \tau_0, \xi_0) \neq 0$ where $\{ , \}$ denotes Poisson bracket. Let us pursue a bicharacteristic strip through $(0, 0, \tau, \xi) \in \pi^{-1}(0) \cap V_R$ where $\pi: \sqrt{-1}S^*M \to M$. It will fall across D. Then two bicharacteristic strips come forth from there and they may again fall across D and so on. We call the union of these bicharacteristic strips the characteristic conoid.

Theorem. *The elementary solution $E_j(t, x)$ is micro-analytic except the characteristic conoid.*

The essential part of the proof of this theorem is the following lemma.

Lemma. *Let $P(t, x, D_t, D_x)$ be a pseudo-differential operator defined in a neighborhood of $(0, \cdots, 0, \sqrt{-1}(0, \cdots, 1)\infty)$ such that $\sigma(P) = t^{s_1}\tau^{s_2}$. If a microfunction u satisfies*

a) $P(t, x, D_t, D_x)u=0$,

b) $u=0$ *on* $\{(t, 0, \cdots, 0, \sqrt{-1}(0, \cdots, 1))/t<0\} \cup \{(0, \cdots, 0, \sqrt{-1}(t, 0, \cdots, 1)\infty / t<0\}$ *or b')* $u=0$ *on* $\{(t, 0, \cdots, 0, \sqrt{-1}(0, \cdots, 1)\infty / t<0\} \cup \{(0, \cdots, 0, \sqrt{-1}(t, 0, \cdots, 1)\infty) / t>0\}$, *then* $u=0$ *at* $(0, \cdots 0, \sqrt{-1}(0, \cdots, 1)\infty)$.

This is an easy corollary of the existence of a good elementary solution of Kashiwara-Kawai.

References

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