60. *Elements of Finite Order in an Ordered Semigroup Whose Product is of Infinite Order*

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We use the terminology and notation in [1] freely. By an *ordered semigroup* we mean a semigroup with a simple order which is compatible with the semigroup operation. Let a be an element of an ordered semigroup S. a is called *positive* [*negative*; *nonnegative*; *nonpositive*] if $a<a^2$ [$a^2<a$; $a\leq a^2$; $a^2\leq a$]. The number of distinct powers of a is called the *order* of a. The semigroup S is called *nonnegatively ordered* if all elements of S are nonnegative.

In [8], we gave the property that the set of all elements of finite order of a nonnegatively ordered semigroup S forms a subsemigroup of S, if it is nonempty. This property does not hold in general in ordered semigroups not necessarily nonnegatively ordered. In fact, Kuroki [2] gave the ordered semigroup K consisting of elements

$$e<x<u_1<u_2<\cdots<r_1<r_2<\cdots$$
$$<g<h<s_1<s_2<\cdots<y<v_1<v_2<\cdots<f$$

with the multiplication table

	e	x	u_j	r_j	g	h	s_j	y	v_j	f
e	e	e	e	e	e	e	e	e	e	e
x	e	e	e	e	e	e	u_j	r_1	r_{j+1}	g
u_i	e	e	e	e	e	e	u_{i+j}	r_{i+1}	r_{i+j+1}	g
r_i	e	u_i	u_{i+j}	r_{i+j}	g	g	g	g	g	g
g	g	g	g	g	g	g	g	g	g	g
h	h	h	h	h	h	h	h	h	h	h
s_i	h	h	h	h	h	h	s_{i+j}	v_i	v_{i+j}	f
y	h	s_1	s_{j+1}	v_j	f	f	f	f	f	f
v_i	h	s_{i+1}	s_{i+j+1}	v_{i+j}	f	f	f	f	f	f
f	f	f	f	f	f	f	f	f	f	f

and the ordered semigroup K' arising from K by identifying the elements g and h, as examples of ordered semigroups in which the elements x and y are elements of finite order but the element $r_1=xy$ is an element of infinite order.

In this paper we consider conversely and prove the following

Theorem. *Let x and y be elements of finite order of an ordered semigroup S such that $x\leq y$, $xy\leq yx$ and xy is a positive element of in-*

finite order. Then the subsemigroup T generated by elements x and y is isomorphic to either one of ordered semigroups K and K'.

Proof. We denote by m and n the orders of elements x and y, respectively. Since xy is positive, we have $xy < xyxy$ and so

(1) $\qquad x < xyx \quad \textit{and} \quad y < yxy.$

Hence $y < yxy \leq y^3$ and so

(2) $\qquad$ *y is positive.*

If x were nonnegative, then by [8] Lemma 4.7, xy would be an element of finite order, contradicting the assumption. Hence

(3) $\qquad$ *x is negative.*

Put $e = x^m$ and $f = y^n$. Then clearly

(4) $\qquad$ *e and f are idempotents.*

For every natural number i, we have $x(yx)^i y = (xy)^{i+1} < (xy)^{i+2} = x(yx)^{i+1} y$ and so $(yx)^i < (yx)^{i+1}$. Hence

(5) $\qquad$ *yx is a positive element of infinite order.*

By way of contradiction, we assume that $y \leq (yx)^i$ for some natural number i. Then $y \leq (yx)^i \leq y^{2i}$ and so y and yx lie in the same archimedean class. This contradicts [6] Theorem 3, since y is an element of finite order and by (5) yx is an element of infinite order. Hence

(6) $\qquad (yx)^i < y \quad$ *for every natural number i.*

By (1) we have $y < y(xy) \leq y(yx) = y^2 x$. Hence $f = y^n \leq y^{n+1} x \leq y^{n+2} \leq f^{n+2} = f$. Hence $f = y^{n+1} x = fx$. Also $fy = y^{n+1} = f$. Hence

(7) $\qquad fw = f \quad$ *for every $w \in T$.*

By (7) $(wf)^2 = wfwf = wf$. Hence

(8) $\qquad$ *wf is an idempotent for every $w \in T$.*

By [4] Corollary of Lemma 1, the set of idempotents of S forms a subsemigroup of S, which we denote by E. By way of contradiction we assume that $yx \leq yef$. Then by (8) yef is an idempotent and so $(yx)^{mn+1} \leq (yef)^{mn+1} = yef$. On the other hand, by (7) and (4) $yef = yefx = yx^{mn} y^{mn} x \leq y(xy)^{mn} x = (yx)^{mn+1}$. Hence we have $yef = (yx)^{mn+1}$. But this is a contradiction, since by (5) yx is an element of infinite order and by (8) yef is an idempotent. Hence we have $yef < yx$ and so $ef < x$. Since $e, f \in E$, we have $ef \in E$. Hence $e = x^m = x^{m+1} \leq x^m y = ey \leq ey^n = ef = (ef)^m \leq x^m = e$ and so $ey = e$. Also $ex = x^{m+1} = x^m = e$. Hence

(9) $\qquad ew = e \quad$ *for every $w \in T$.*

By (7) and (9) $ef = e$ and $fe = f$ and so $e \mathcal{L} f$ in the semigroup E. Also by (2) and (3) $e = x^m < x < y < y^n = f$. Hence by [8] Lemma 1.13 and its dual we have

(10) $\qquad m = n = 2.$

By (1) $y < yxy \leq yxy^2 = yxf$. Hence $f = y^2 \leq (yxf)^2 = yxf \leq y^2 f = f$ and so

(11) $\qquad yxf = f.$

By (6) $xye = xyx^2 \leq xyx \leq xy$. But by (9) xye is an idempotent and by assumption xy is an element of infinite order. Hence $xy > xye = xyey$ by (9). Therefore $xye < x$. Hence $e = x^2e \leq xye = (xye)^2 \leq x^2 = e$ and so

(12) $$xye = e.$$

Since xy and yx are elements of infinite order, we have $(xy)^i xy = (xy)^{i+1} < (xy)^{i+2} = (xy)^{i+1}xy$ and $(yx)^i yx = (yx)^{i+1} < (yx)^{i+2} = (yx)^{i+1}yx$. Hence

(13) $$(xy)^i x < (xy)^{i+1}x \quad \textit{and} \quad (yx)^i y < (yx)^{i+1}y$$ *for every natural number i.*

By (12) and (1) we have $(xy)^i x^2 = (xy)^i e = e < x < xyx$. Hence

(14) $$(xy)^i x < xy \qquad \textit{for every natural number } i.$$

By (5) and (7) we have $(yx)^i yx = (yx)^{i+1} < f = fx$. Hence

(15) $$(yx)^i y < f \qquad \textit{for every natural number } i.$$

Put $h = ye$. Then by (9) h is an idempotent. Also $x(xf) = ef = e < xy$ and so $xf < y$. Hence $xf = xfe \leq ye$. Thus

(16) $$g \leq h.$$

Put $u_i = (xy)^i x$, $r_i = (xy)^i$, $s_i = (yx)^i$ and $v_i = (yx)^i y$. Now it is easy to check the conclusion of the theorem.

Remark. It is easily seen that four idempotents e, f, g and h lie in the same $\mathcal{L}$-class in the semigroup E and $\{e, g\}$ and $\{h, f\}$ are consecutive pairs of elements on the $\mathcal{L}$-class.

References

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