59. On a Problem of Fossum

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Robert M. Fossum proposes the following problem in his book "The divisor class group of a Krull domain":*)

Problem. Let k be a field of characteristic not equal to 2, and $F(X_1, X_2, X_3, X_4)$ a non-degenerate quadratic form over k. Find a necessary and sufficient condition in order that $A_F = K[X_1, X_2, X_3, X_4]/(F)$ may be a factorial ring.

The purpose of the present note is to give the answer of the problem. In this note, we employ the same terminology and notation as of [F].

Lemma 1. *Let* $F = X_1^2 + aX_2^2 + bX_3^2 + cX_4^2$ $(abc \neq 0,\ a, b, c \in k)$. *If* $c = ab$ *then* $\mathrm{Cl}(A_F) \simeq \boldsymbol{Z}$

Proof. If $t = \sqrt{-a}$ is in k, then $F = UV + YZ$ with $U = X_1 + tX_2$, $V = X_1 - tX_2$, $Y = b(X_3 + tX_4)$, $Z = X_3 - tX_4$ and therefore the assertion in this case is obvious by [F], § 14.

If t is not in k, then we can show that class $\mathfrak{p}$, where $\mathfrak{p} = (x_1^2 + ax_2^2, x_3^2 + ax_4^2, x_1x_3 + ax_2x_4, x_1x_4 - x_2x_3)$ and x_i is the image of X_i in A_F $(i = 1, 2, 3, 4)$, generates infinite cyclic group. Since we know that $\mathrm{Cl}(A_F)$ is a subgroup of an infinite cyclic group by the proof of Klein-Nagata theorem, we deduce that $\mathrm{Cl}(A_F) \simeq \boldsymbol{Z}$. (Cf. the proof of L. Roberts quoted in [F], § 11 p. 52.)

Lemma 2. *Let* $F = X_1^2 + aX_2^2 + bX_3^2 + cX_4^2$ $(a, b, c \in k,\ abc \neq 0)$. *If none of* $-a, -bc, abc$ *is the square of any element of* k, *then* A_F *is factorial.*

Proof. In view of the proof of Klein-Nagata theorem, it is sufficient to show that $G = bX_3^2 + cX_4^2$ is irreducible in $k(t)[X_3, X_4]$, where $t = \sqrt{-a}$. To do this it is sufficient to prove that $-c/b$ cannot be written as the square of any element of $k(t)$. Assume the contrary, i.e., that it holds that

$$-c/b = (\alpha + \beta t)^2 = \alpha^2 - \beta^2 a + 2\alpha\beta t \qquad (\alpha, \beta \in k)$$

Since 1 and t are linearly independent over k we must have $2\alpha\beta = 0$. Since we assumed that ch $k \neq 2$ and since $-c/b$ is not the square of any element of k, we have $\beta \neq 0$ and therefore $\alpha = 0$. But then

$$-c/b = -a\beta^2 \quad \text{and} \quad abc = \beta^2 a^2 b^2,$$

*) In this note, the symbol [F] will refer to this literature, Ergebn. Math. Bd. 74, Springer (1973).

a contradiction. This completes the proof of Lemma 2.

It is well known that a non-degenerate quadratic form F in $k[X_1, X_2, X_3, X_4]$ can be written as

$$F(X_1, X_2, X_3, X_4)=X_1^2+aX_2^2+bX_3^2+cX_4^2 \qquad (a, b, c \in k\ abc\neq 0)$$

by an adequate linear transformation. In this case, if $-b$ or $-c$ is the square of an element of k, then by changing indices of X we may assume that $-a$ is the square of the element of k; we may do the same even if some one of $-ab$, $-bc$, $-ca$ is the square of an element of k by multiplying an element of k and by changing indices of X. (For example, if $-ab=\beta^2, \beta\in k$ then $\frac{1}{a}F=\frac{1}{a}X_1^2+X_2^2-\frac{\beta^2}{a^2}X_3^2+\frac{c}{a}X_4^2$ and we put $X_1'=X_2, X_2'=X_3, X_3'=X_1, X_4'=X_4$). Therefore the following theorem covers all the cases.

Theorem. *Let* $F=X_1^2+aX_2^2+bX_3^2+cX_4^2\ (abc\neq 0)$ *and* $A_F=k[X_1, X_2, X_3, X_4]/(F)$.

1) *If* $-a$ *is the square of an element of* k, *then* A_F *is factorial if and only if* $-bc$ *is not the square of any element of* k.

2) *If none of* $-a, -b, -c, -ab, -bc, -ca$ *is the square of any element of* k, *then* A_F *is factorial if and only if* abc *is not the square of any element of* k.

3) *If* A_F *is not factorial, then* $\mathrm{Cl}\,(A_F)\simeq \boldsymbol{Z}$.

Proof. In the case 1), $G=bX_3^2+cX_4^2$ is irreducible if and only if $-bc$ is not the square of an element of k, and we prove this case.

In the case 2), if abc is not the square of any element of k, then A_F is factorial by Lemma 2.

In abc is the square of an element of k, then

$$abc=\alpha^2\ (\alpha\in k), \qquad c=\frac{\alpha^2}{a^2b^2}\cdot ab$$

and by putting $X_4'=\frac{\alpha}{ab}X_4$, it holds that

$$F=X_1^2+aX_2^2+bX_3^2+abX_4'^2$$

and we have $\mathrm{Cl}\,(A_F)\simeq \boldsymbol{Z}$ by Lemma 1.

The non-factorial case of 1) is also the same as the proof of Lemma 1. This completes the proof.