# 75. *On Hodge Structure of Isolated Singularity of Complex Hypersurface*

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**Introduction.** The Hodge spectral sequence for an isolated singularity of (complex) analytic space is defined as follows. Note first that, given a complex manifold $Z$, the bigrading of differential forms of $Z$ together with the operators $\partial$ and $\bar{\partial}$ defines a double complex. The Hodge structure $(E_r^{p,q}(Z), d_r)$ of $Z$ is the spectral sequence of this double complex so chosen that $E_1^{p,q}(Z)=H^q(Z, \Omega_Z^p)$ where $\Omega_Z^p$ denotes the sheaf of holomorphic $p$-forms on $Z$. Let now $(X, x)$ denote the situation where $x$ is an isolated singular point of an analytic space $X$. For sufficiently small neighborhood $U$ of $x$, $(E_r^{p,q}(U\backslash x), d_r)$ are well defined and form a direct system with the restriction maps. Set

$$E_r^{p,q}(X, x)=\lim_{\overrightarrow{U}} E_r^{p,q}(U\backslash x).$$

The map $d_r: E_r^{p,q}(X, x)\to E_r^{p+r,q-r+1}(X, x)$ is naturally induced. $(E_r^{p,q}(X, x), d_r)$ thus obtained is the Hodge spectral sequence of the isolated singularity $(X, x)$. If $X$ is $n$-dimensional, then $E_r^{p,n}(X, x)=0$ by Malgrange [3]. By Andreotti-Grauert [1] $E_1^{p,q}(X, x)$ are finite-dimensional (over $\boldsymbol{C}$) if $1\leqq q\leqq n-2$.

The main result is the following

**Theorem 1.** *Let $n\geqq 3$ and suppose $(X, x)$ is a hypersurface singularity, that is, there is a holomorphic function $f$ in a domain $Y$ of $\boldsymbol{C}^{n+1}: (z_0, \cdots, z_n)$ such that $X=\{z\in Y; f(z)=f(x)\}$, and such that $\partial f(z)/\partial z_i=0$ $(0\leqq i\leqq n)$ if and only if $z=x$. Let $E_r^{p,q}(X, x)$ be denoted for short by $E_r^{p,q}$. Then the following conclusions are valid.*

(i) $E_1^{p,q}=0$ *if* $q\neq 0$, $q\neq n-1$, $p+q\neq n-1$, $p+q\neq n$.

(ii) *There are canonical isomorphisms:*

$$E_1^{2,n-2}\cong E_1^{3,n-3}\cong\cdots\cong E_1^{n-1,1}$$
$$E_1^{1,n-2}\cong E_1^{2,n-3}\cong\cdots\cong E_1^{n-2,1}$$

(ii)′ $\dim E_1^{n-q,q-1}=\dim E_1^{n-q,q}$ *for* $2\leqq q\leqq n-2$

(iii) $E_2^{p,q}$ *are all finite-dimensional.*

(iv) $E_2^{p,0}=0$ *for* $1\leqq p\leqq n-2$.

(iv)′ $E_2^{p,n-1}=0$ *for* $2\leqq p\leqq n-1$.

(v) *If $\mu$ is the multiplicity of the hypersurface singularity $(X, x)$ in the sense of Milnor* [4], *then*

$$(*)\qquad \begin{aligned}\mu&=\dim E_1^{n-1,1}+\dim E_2^{n,0}-\dim E_2^{n-1,0}\\ &=\dim E_1^{1,n-2}+\dim E_2^{0,n-1}-\dim E_2^{1,n-1}.\end{aligned}$$

The formula for the monodromy is obtained only in case $f$ is quasi-homogeneous, that is, $f$ can be written in the form

$$f(z)=\sum_{a_0i_0+\cdots+a_ni_n=m} c_{i_0\cdots i_n} z_0^{i_0}\cdots z_n^{i_n}$$

where $a_0,\cdots,a_n,m>0$ are all integers. In this case the maps $h_\varphi(z)=(e^{2\pi i a_0\varphi}z_0,\cdots,e^{2\pi i a_n\varphi}z_n)$ induce an $\boldsymbol{R}/\boldsymbol{Z}$-action on $(X,x)$. In particular $h_{1/m}$ induces an endomorphism of $E_1^{n-1,1}$. Denote by $\varDelta'(t)$ the characteristic polynomial of this endomorphism. Then the characteristic polynomial $\varDelta(t)$ of the monodromy of $(X,x)$ is given by

$$(**)\qquad \varDelta(t)=(t-1)^\nu\varDelta'(t)$$

where $\nu=\dim E_2^{n,0}-\dim E_2^{n-1,0}$.

**1. Sketch of the proof.** Let $f$, $Y$, $(X,x)$ be as in Theorem 1. We denote by $\Omega^p$ the sheaf holomorphic $p$-forms on $Y$. Following Brieskorn [2] we set

$$\Omega_f^p=\Omega^p/df\wedge\Omega^{p-1}.$$

Then $\Omega_f^\cdot$ is naturally a complex of sheaves. For an open ball $B$ in $\boldsymbol{C}^{n+1}$ with center at $x$, we set $B_*=B\backslash x$ and set

$$H_*^q(\Omega_f^p)=\varinjlim_B H^q(B_*,\Omega_f^p).$$

Consider the exact sequence

$$0\longrightarrow\Omega_f^{p-1}\xrightarrow{df}\Omega^p\longrightarrow\Omega_f^p\longrightarrow 0$$

where the first map is induced by the exterior multiplication of $df$. Using long exact sequence associated with this, we obtain at first

**Lemma 1.** $H_*^q(\Omega_f^p)=0$ *if* $q\neq 0$, $q\neq n$, $p+q\neq n$.
$H_*^q(\Omega_f^p)\cong H_*^{q+1}(\Omega_f^{p-1})$ *if* $1\leqq q\leqq n-2$.

Combining this lemma with the Hartogs-Osgood theorem, and using the crucial parts of the long exact sequences, we obtain the isomorphisms

$$(1)\qquad H^p(\Omega_{f,x}^\cdot)\cong H^p(H_*^0(\Omega_f^\cdot))\qquad p\leqq n-1$$

and the exact sequence

$$(2)\qquad 0\rightarrow H'^n(\Omega_{f,x}^\cdot)\rightarrow H^n(H_*^0(\Omega_f^\cdot))\rightarrow H_*^1(\Omega_f^{n-1})\rightarrow 0$$

where $H'^n(\Omega_{f,x}^\cdot)$ is the notation of [2]. These (1) and (2) together with [2] implies that

$$(3)\qquad H^p(H_*^0(\Omega_f^\cdot))=0\qquad 1\leqq p\leqq n-1$$

and that Ker $(\alpha)$, Cok $(\alpha)$ is finite-dimensional and

$$(4)\qquad \mu=\dim \operatorname{Cok}(\alpha)-\dim\operatorname{Ker}(\alpha)$$

where $\alpha: H^n(H_*^0(\Omega_f^\cdot))\rightarrow H^n(H_*^0(\Omega_f^\cdot))$ is the map induced by the multiplication of $f$ in $\Omega_f^\cdot$.

Consider now the exact sequence

$$0\longrightarrow\Omega_f^p\xrightarrow{f}\Omega_f^p\longrightarrow\Omega_f^p/f\longrightarrow 0$$

where $\Omega_f^p/f$ abbreviates $\Omega_f^p/f\Omega_f^p$. Using the associated long exact sequences, with Lemma 1 in mind, we can prove (i), (ii) and (ii)$'$. By the crucial parts of these sequences, we obtain also the following three exact sequences:

$$(5)\qquad H^1_*(\Omega_f^{n-1})\to H^1_*(\Omega_f^{n-1})\to E_1^{n-1,1}\to 0$$

$$(6)\qquad 0\to H^0(H^0_*(\Omega^\cdot_f))\to H^0(H^0_*(\Omega^\cdot_f))\to E_2^{0,0}\to\cdots$$
$$\cdots\to H^{n-1}(H^0_*(\Omega^\cdot_f))\to H^{n-1}(H^0_*(\Omega^\cdot_f))\to E_2^{n-1,0}$$

$$(7)\qquad 0\to \mathrm{Ker}\,(\alpha)\to E_2^{n-1,0}\to K\to \mathrm{Cok}\,(\alpha)\to E_2^{n,0}\to 0$$

where $K=\mathrm{Ker}\,(H^1_*(\Omega_f^{n-1})\to H^1_*(\Omega_f^{n-1}))$. Combining (3) and (6) we prove (iv). Using (4), (5) and (7) we prove (iii) and obtain the formula

$$\mu=\dim E_1^{n-1,1}+\dim E_2^{n,0}-\dim E_2^{n-1,0}.$$

Now the full formula (*) follows from this by the Poincaré duality; (iv)′ follows from (iv) also by the Poincaré duality. The proof of formula (**) is almost evident from the course of the proof of (*).

The details will be published elsewhere.

## References

[1] Andreotti, A., and Grauert, H.: Théorèmes de finitude pour la cohomologie des espaces complexes. Bull. Soc. Math. France, **90**, 193–259 (1962).

[2] Brieskorn, E.: Die Monodromie der isolierten Singularitäten von Hyperflächen. Manuscripta Math., **2**, 103–161 (1970).

[3] Malgrange, B.: Faisceaux sur des variétés analytiques réelles. Bull. Soc. Math. France, **85**, 231–237 (1957).

[4] Milnor, J.: Singular points of complex hypersurfaces. Ann. of Math. Studies Number 61, Princeton; Princeton Univ. Press (1968).