108. On Common Fixed Point Theorems of Mappings

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In his recent book [1], V. I. Istrăţesçu proved some common fixed point theorems about contraction mappings. In this paper, we shall generalize his results.

Let (X, ρ) be a complete metric space, and T_k $(k=1, 2, \cdots, n)$ a family of mappings of X into itself.

Theorem 1. *If T_k $(k=1, 2, \cdots, n)$ satisfies*

1) $T_k T_l = T_l T_k$ $(k, l=1, 2, \cdots, n)$,

2) *There is a system of positive integers $m_1, m_2, \cdots, m_n$ such that*

$$\begin{aligned} \rho(T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}x, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}y) \\ \le \alpha\rho(x, y) + \beta[\rho(x, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}x) \\ + \rho(y, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}y)] + \gamma[\rho(x, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}y) \\ + \rho(y, T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}x)] \end{aligned} \tag{1}$$

for every x, y of X, where α, β, γ are non-negative and $\alpha+2\beta+2\gamma<1$, then T_k $(k=1, 2\cdots, n)$ have a unique common fixed point.

Proof. To prove Theorem, we use I. Rus theorem [2]. Let $U = T_1^{m_1}T_2^{m_2}\cdots T_n^{m_n}$, then by (1), we have

$$\begin{aligned} \rho(Ux, Uy) \le \alpha(x, y) + \beta[\rho(x, Ux) + \rho(y, Uy)] \\ + \gamma[\rho(x, Uy) + \rho(y, Ux)] \end{aligned}$$

for all x, y of X. Hence by I. Rus theorem, U has a unique fixed point ξ in X. Therefore $U\xi=\xi$, then we have

$$T_i(U\xi) = T_i\xi \qquad (i=1, 2, \cdots, n). \tag{2}$$

By the commutativity of $\{T_k\}$, (2) implies

$$U(T_i\xi) = T_i\xi.$$

Since U has a unique fixed point ξ, we obtain $T_i\xi=\xi$ $(i=1, 2, \cdots, n)$. Hence ξ is a common fixed point of the family $\{T_k\}$.

Let ξ, η be common fixed points of $\{T_k\}$, then by (1), we have

$$\begin{aligned} \rho(\xi, \eta) = \rho(U\xi, U\eta) \le \alpha\rho(\xi, \eta) \\ + \beta[\rho(\xi, U\xi) + \rho(\eta, U\eta)] + \gamma[\rho(\xi, U\eta) + \rho(\eta, U\xi)], \end{aligned}$$

which implies

$$\rho(\xi, \eta) \le \alpha\rho(\xi, \eta) + 2\gamma\rho(\xi, \eta).$$

From $\alpha+2\gamma<1$, we have $\rho(\xi, \eta)=0$, i.e. $\xi=\eta$. We have the uniqueness, and we complete the proof.

Theorem 2. *If $\{T_k\}$ satisfies the conditions:*

1) *$T_1T_2\cdots T_n$ commutes with every T_i,*

2) *for every x, y of X,*

$$\rho(T_1T_2\cdots T_nx, T_nT_{n-1}\cdots T_1y)\le\alpha\rho(x,y)+\beta[\rho(x,T_1T_2\cdots T_nx)+\rho(y,T_nT_{n-1}\cdots T_1y)]+\gamma[\rho(x,T_nT_{n-1}\cdots T_1y)+\rho(y,T_1T_2\cdots T_nx)], \tag{3}$$

where α,β,γ *are non-negative, and* $\alpha+2\beta+2\gamma<1$, *then* T_k $(k=1,2,\cdots,n)$ *have a unique common fixed point.*

Proof. Let $U=T_1T_2\cdots T_n$, $V=T_nT_{n-1}\cdots T_1$, then by (3), we have

$$\rho(Ux,Vy)\le\alpha\rho(x,y)+\beta[\rho(x,Ux)+\rho(y,Vy)]+\gamma[\rho(x,Vy)+\rho(y,Ux)] \tag{4}$$

for all x, y of X. By I. Rus theorem [2], U and V have a unique common fixed point ξ. Then $U\xi=V\xi=\xi$.

For any T_i, $T_i(U\xi)=T_i\xi$. By the assumption, $U(T_i\xi)=T_i\xi$. $T_i\xi$ is a fixed point of U, and ξ is a fixed point of V. By the relation (φ), we have

$$\rho(T_i\xi,\xi)\le\alpha\rho(T_i\xi,\xi)+2\gamma\rho(T_i\xi,\xi).$$

Hence $T_i\xi=\xi$ $(i=1,2,\cdots,n)$, which means that ξ is a common fixed point of $\{T_k\}$. It is easily seen that ξ is a unique common fixed point of $\{T_k\}$. This completes the proof.

Remark 1. In Theorems 1, 2, if $\alpha=\gamma=0$, then we obtain Istrătesçu results (see [1], pp. 100-105).

References

[1] V. I. Istrătescu: Introducere in teoria punctelor fixe. Bucarest (1973).
[2] I. A. Rus: On common fixed points. Studia Universitatis Babes-Bolyai, fasc., **1**, 31–33 (1973).