104. Localization of G-spaces

By Yoshiharu MATAGA
Takamatsu Technical College

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1. Introduction. In [1] D. H. Gottlieb has introduced the notion of G-spaces. The purpose of this note is to apply the localization theory to G-spaces. A space X is called a G-space if, when we fix $x_0 \in X$ arbitrarily as the base point, for any integer m and for any element $\alpha \in \pi_m(X, x_0)$ there exists a map $F: (X \times S^m, (x_0, s_0)) \to (X, x_0)$ such that $F_{|X \times \{s_0\}}: X \to X$ is the identity map and $F_{|\{x_0\} \times S^m}: (S^m, s_0) \to (X, x_0)$ represents α, where $s_0 \in S^m$ is the base point.

In [4] it has been proved that for any 1-connected CW-complex X of finite type and any set $\boldsymbol{P}$ of primes there exist the localized space $X_{\boldsymbol{P}}$ which is a 1-connected, countable CW-complex, and the localization map $j_X: X \to X_{\boldsymbol{P}}$ (i.e. the induced map $(j_X)_*$ localizes the homology group with integer coefficient and the homotopy groups with respect to $\boldsymbol{P}$), and moreover that $X_{\boldsymbol{P}}$ is determined up to homotopy by the homotopy type of X and by the set $\boldsymbol{P}$.

When $\boldsymbol{P}$ consists of one element p, we denotes $X_{\boldsymbol{P}} = X_{(p)}$.

The main theorem of this note is the next one.

Theorem 1. *Let X be a 1-connected, finite CW-complex. Then X is a G-space if and only if $X_{(p)}$ is a G-space for all primes p.*

2. Proof of Theorem 1. An m-th evaluation subgroup, denoted by $G_m(X, x_0)$, of the homotopy group $\pi_m(X, x_0)$ is the set of all $\alpha \in \pi_m(X, x_0)$ for which there exist a map $F: (X \times S^m, (x_0, s_0)) \to (X, x_0)$ and a representative $f: (S^m, s_0) \to (X, x_0)$ of α such that $F|_{X \times \{s_0\}} = \text{identity}$ and $F|_{\{x_0\} \times S^m} = f$. In fact $G_m(X, x_0)$ is a subgroup of $\pi_m(X, x_0)$ [1; § 1]. Note that X is a G-space if and only if $G_m(X, x_0) = \pi_m(X, x_0)$ for arbitrary point $x_0 \in X$ and for all m.

Let $\mathcal{C}_{\boldsymbol{P}}$ be a Serre class of finite abelian groups whose orders are prime to p for all $p \in \boldsymbol{P}$, where $\boldsymbol{P}$ is a set of primes.

According to H. B. Haslam [2] we call a 1-connected space X a mod $\boldsymbol{P}$ G-space if $\pi_m(X, x_0)/G_m(X, x_0) \in \mathcal{C}_{\boldsymbol{P}}$ for arbitrary point $x_0 \in X$ and for all m.

Lemma 2 [1; 1–3]. (1) *Let $x_0, x_1 \in X$ and let $\sigma: I \to X$ be a path in X such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Then the induced isomorphism $\sigma_*: \pi_m(X, x_1) \cong \pi_m(X, x_0)$ gives the isomorphism $G_m(X, x_1) \cong G_m(X, x_0)$.*

(2) *Let $x_0 \in X$ and $y_0 \in Y$ and let $f: (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Suppose x_0 is closed in X and y_0 closed in Y and (X, x_0)*

and (Y, y_0) have the homotopy extension property. Then the induced isomorphism $f_*: \pi_m(X, x_0) \cong \pi_m(Y, y_0)$ gives the isomorphism $G_m(X, x_0) \cong G_m(Y, y_0)$.

Theorem 3. *Let X be a 1-connected, finite CW-complex and $\boldsymbol{P}$ be a set of primes. Then X is a* mod $\boldsymbol{P}$ *G-space if and only if $X_{\boldsymbol{P}}$ is a G-space.*

We will show Theorem 1, assuming Theorem 3.

Proof of Theorem 1. Suppose X is a G-space. Then clearly X is a mod p G-space for all primes p. It follows from Theorem 3 that $X_{(p)}$ is a G-space for all primes p.

Conversely suppose $X_{(p)}$ is a G-space for all primes p. Then from Theorem 3 it follows that X is a mod p G-space for all primes p, that is $\pi_m(X, x_0)/G_m(X, x_0) \in \mathcal{C}_p$ for all primes p, where x_0 is the base point chosen arbitrarily. Therefore $G_m(X, x_0) = \pi_m(X, x_0)$. Since m is arbitrary, this concludes that X is a G-space. Q.E.D.

To prove Theorem 3 some lemmas will be needed. Let $\mathcal{H}_1$ be a category of 1-connected, finite CW-complexes. In [4] the localization of $X \in \mathcal{FC}_1$ at $\boldsymbol{P}$ is constructed as the union of a $\boldsymbol{P}$-sequence $\{X_i, g_i\}_{i=0,1,\dots}$ of X, where $X_0 = X$, $X_i \in \mathcal{FC}_1$ $(i \geq 0)$ and $g_i: X_{i-1} \to X_i$ is a $\boldsymbol{P}$-equivalence, that is g_i induces isomorphisms $g_{i*}: H_*(X_{i-1}; \boldsymbol{Z}_p) = H_*(X_i; \boldsymbol{Z}_p)$ for all $p \in \boldsymbol{P}$. As for the definition of a $\boldsymbol{P}$-sequence and its existence for any $X \in \mathcal{FC}_1$ and any $\boldsymbol{P}$ we refer to [4].

Lemma 4. *Let $X \in \mathcal{FC}_1$. Then $(j_X)_*: \pi_m(X, x_0) \to \pi_m(X_{\boldsymbol{P}}, \bar{x}_0)$ carries $G_m(X, x_0)$ into $G_m(X_{\boldsymbol{P}}, \bar{x}_0)$, where $\bar{x}_0 = j_X(x_0)$.*

Proof. Let $\{X_i, g_i\}$ be a $\boldsymbol{P}$-sequence of X. We may assume that $g_i: X_{i-1} \to X_i$ is an inclusion of a subcomplex. So we may also assume that $g_i \times id: X_{i-1} \times S^m \to X_i \times S^m$ is an inclusion of a subcomplex. Choose the base points $x_i \in X_i$ so that $x_i = g_i(x_{i-1})$ $(i = 1, 2, \cdots)$.

Let $F: (X \times S^m, (x_0, s_0)) \to (X, x_0)$ be a map such that $F|_{X \times \{s_0\}} =$ identity and $F|_{\{x_0\} \times S^m}$ represents $\alpha \in \pi_m(X, x_0)$. By the similar method to the proof of [4; 1.7] we can find a sequence $\{F_i\}_{i=0,1,\dots}$ of maps, where $F_i: (X_i \times S^m, (x_i, s_0)) \to (X_{\rho(i)}, x_{\rho(i)})$ for some $\rho(i) > i$, such that $F_0 = F$ and the following diagram is homotopy commutative

$$\begin{array}{ccc} X_{i-1} \times S^m & \xrightarrow{\quad g_i \times id \quad} & X_i \times S^m \\ \Big\downarrow F_{i-1} & & \Big\downarrow F_i \\ X_{\rho(i-1)} & \xrightarrow{\quad g_{\rho(i)} \circ \cdots \circ g_{\rho(i-1)+1} \quad} & X_{\rho(i)}. \end{array}$$

Then it is clear that there exists a map $\bar{F}: \bigcup_{i=0}^{\infty} (X_i \times S^m) = X_{\boldsymbol{P}} \times S^m \to \bigcup_{i=0}^{\infty} X_i = X_{\boldsymbol{P}}$ such that $\bar{F} \circ (j_X \times id)$ is homotopic to $j_X \circ F$. Since $F|_{X \times \{s_0\}} =$ identity, it follows from [4; 1.7] that $\bar{F}|_{X_{\boldsymbol{P}} \times \{s_0\}}$ is homotopic to the identity map of $X_{\boldsymbol{P}}$.

Since $(X_{\boldsymbol{P}} \times S^m, X_{\boldsymbol{P}} \times \{s_0\} \cup \{x_0\} \times S^m)$ has the homotopy extension property, there exists a map $G: (X_{\boldsymbol{P}} \times S^m, (\bar{x}_0, s_0)) \to (X_{\boldsymbol{P}}, \bar{x}_0)$ homotopic to $\bar{F}$ such that $G|_{X_P\times\{s_0\}}=$identity. Then clearly $G|_{\{\bar{x}_0\}\times S^m}$ represents $(j_X)_*(\alpha)$. Therefore $(j_X)_*(\alpha)\in G_m(X,\bar{x}_0)$. Q.E.D.

For spaces X and Y, $X\simeq Y$ means that X is homotopy equivalent to Y.

Lemma 5. *Let $X\in\mathscr{F}\mathcal{C}_1$. If X is a* mod $\boldsymbol{P}$ *G-space, there exists a $\boldsymbol{P}$-sequence $\{X_i, g_i\}$ of X such that $X_i\simeq X$ for all i.*

Proof. Since $\pi_m(X,x_0)/G_m(X,x_0)$ is a finite abelian group for all m, it follows from [2; Theorem 1] that X is a mod 0 H-space, that is there exists a multiplication $\mu: X\times X\to X$ such that $\mu\cdot i_j: X\to X$ $(j=1,2)$ are rational equivalences, where $j_j: X\to X\times X$ is the canonical inclusion into the j-th coordinate. By [3; 1.4] a mod 0 H-space is $\boldsymbol{P}$-universal. By [4; 5.3] a $\boldsymbol{P}$-universal space has a required $\boldsymbol{P}$-sequence. Q.E.D.

As for a Moore-Postnikov factorization $\{p_n, E_n, f_n\}$ of a map $f: X\to Y$ we refer to [5; Chap. 8, Sec. 3], where $p_n: E_n\to E_{n-1}$ $(n\geq 1)$ and $f_n: X\to E_n$ $(n\geq 0)$. It is well known that if X and Y are CW-complexes, for all n E_n may satisfy the conditions (i) E_n has the homotopy type of a CW-complex, (ii) e_n, the base point of E_n, is closed in E_n, (iii) (E_n, e_n) has the homotopy extension property.

Lemma 6. *Let $X\in\mathscr{F}\mathcal{C}_1$. Let $\{p_n, E_n, f_n\}$ be a Moore-Postnikov factorization of $f: X\to Y$. If $\pi_m(E^n, e_n)/G_m(E_n, e_n)\in\mathcal{C}_P$ for all m and n, then X is a* mod $\boldsymbol{P}$ *G-space.*

Proof. The proof is similar to that of [2; Proposition 2]. Q.E.D.

Suppose we are given maps $F: (X\times S^m, (x_0, s_0))\to(X, x_0)$ with $F|_{X\times\{s_0\}}=$identity and $f: X\to K(\pi, n+1)$, where π is an abelian group and $n\geq 1$. Let $\mu\in H^{n+1}(X;\pi)$ be the image of the characteristic class $\iota\in H^{n+1}(\pi, n+1;\pi)$ by $f^*: H^{n+1}(\pi, n+1;\pi)\to H^{n+1}(X;\pi)$. By the Künneth theorem $H^*(X\times S^m;\pi)\cong H^*(X;\pi)\otimes H^*(S^m; Z)$. So we may represent $F^*(\mu)=\mu\otimes 1+\nu\otimes\lambda\in H^{n+1}(X\times S^m;\pi)$, where $\lambda\in H^m(S^m; Z)$ is the fundamentalclass and $\nu\in H^{n+1-m}(X;\pi)$. Since ν is determined by μ and the homotopy class of F, we denote it by μF.

Lemma 7 [1; 6–3]. *Let $p: E\to X$ be a principal fibration induced by $f: X\to K(\pi, n+1)$ $(n\geq 1)$, where X has the homotopy type of a 1-connected CW-complex, x_0, the base point of X, closed in X and (X, x_0) has the homotopy extension property. Then there exists a map $G: (E\times S^m, (e_0, s_0))\to(E, e_0)$ such that $G|_{E\times\{s_0\}}=$identity and the diagram*

$$\begin{array}{ccc} E\times S^m & \overset{G}{\dashrightarrow} & E \\ {\scriptstyle p\times id}\downarrow & & \downarrow{\scriptstyle p} \\ X\times S^m & \underset{F}{\longrightarrow} & X \end{array}$$

is homotopy commutative if and only if $\mu F=0$.

Lemma 8. *Let $X\in\mathscr{F}\mathcal{C}_1$. Let $\{p_n, E_n, (j_X)_n\}$ be a Moore-Postnikov factorization of the map $j_X: X\to X_P$. If $\pi_m(E_n, e_n)/G_m(E_n, e_n)\in\mathcal{C}_P$ for all m, then $\pi_m(E_{n+1}, e_{n+1})/G_m(E_{n+1}, e_{n+1})\in\mathcal{C}_P$ for all m.*

Proof. Note that for each i $\pi_i(X_P, X)$ consists only of elements whose orders are finite and prime to all $p \in \boldsymbol{P}$, and that for $i \leq n$ $\pi_i(E_n, e_n)$ and $H_i(E_n; \boldsymbol{Z})$ are finitely generated. From now on π stands for $\pi_{n+1}(X_P, X)$. So $p_{n+1}: E_{n+1} \to E_n$ is a principal fibration induced by some map $f: E_n \to K(\pi, n+1)$. Let $F: (E_n \times S^m, (e_n, s_0)) \to (E_n, e_n)$ be a map such that $F|_{E_n \times \{s_0\}} = \text{identity}$.

(i) Suppose $1 < m \leq n+1$ and $m \neq n$. Since $n+1-m < n$, $H^{n+1-m}(E_n; \pi)$ is a torsion group whose elements have orders prime to p for all $p \in \boldsymbol{P}$. Let q be the order of μF, where $(q, p) = 1$ for all $p \in \boldsymbol{P}$. Let $g: (S^m, s_0) \to (S^m, s_0)$ bea map of degree q. Then it is clear that for the map $F \circ (id \times g): E_n \times S^m \to E_n \times S^m \to E_n$ there holds $\mu(F \circ (id \times g)) = 0$. By Lemma 7 there exists a map $G: E_{n+1} \times S^m \to E_{n+1}$ such that $G|_{E_{n+1} \times \{s_0\}} = \text{identity}$ and the diagram

$$\begin{array}{ccccc} E_{n+1} \times S^m & & \xrightarrow{\quad G \quad} & & E_{n+1} \\ \downarrow{\scriptstyle p_{n+1} \times id} & & & & \downarrow{\scriptstyle p_{n+1}} \\ E_n \times S^m & \xrightarrow[id \times g]{} & E_n \times S^m & \xrightarrow[F]{} & E_n \end{array}$$

is homotopy commutative. Since $(p_{n+1})_*: \pi_m(E_{n+1}, e_{n+1}) \to \pi_m(E_n, e_n)$ is a monomorphism, the above fact implies that if $(p_{n+1})_*(\beta) \in G_m(E_n, e_n)$ for $\beta \in \pi_m(E_{n+1}, e_{n+1})$, there exists an integer q with $(q, p) = 1$ for all $p \in \boldsymbol{P}$ such that $q\beta \in G_m(E_{n+1}, e_{n+1})$. Thus $(p_{n+1})_*^{-1}(G_m(E_n, e_n))/G_m(E_{n+1}, e_{n+1}) \cap (p_{n+1})_*^{-1}(G_m(E_n, e_n)) \in \mathcal{C}_P$. From the assumption $\pi_m(E_n, e_n)/G_m(E_n, e_n) \in \mathcal{C}_P$, it follows that $\pi_m(E_{n+1}, e_{n+1})/(p_{n+1})_*^{-1}(G_m(E_n, e_n)) \in \mathcal{C}_P$. Therefore $\pi_m(E_{n+1}, e_{n+1})/G_m(E_{n+1}, e_{n+1}) \in \mathcal{C}_P$.

(ii) Suppose $m = n$. From the homotopy exact sequence of the fibration $p_{n+1}: E_{n+1} \to E_n$ it follows that $(p_{n+1})_*: \pi_n(E_{n+1}, e_{n+1}) \to \pi_n(E_n, e_n)$ is an epimorphism and that $\operatorname{Ker}(p_{n+1})_*$ is a torsion group whose elements have orders prime to all $p \in \boldsymbol{P}$. Furthermore since $\pi_n(E_{n+1}, e_{n+1})$ is finitely generated, $\operatorname{Ker}(p_{n+1})_*$ is a finite group. Let q' be the order of $\operatorname{Ker}(p_{n+1})_*$. Since E_n is 1-connected, $H^{n+1-m}(E_n; \pi) = H^1(E_n; \pi) = 0$. So $\mu F = 0$. It follows from Lemma 7 that there exists a map $G: (E_{n+1} \times S^m, (e_{n+1}, s_0)) \to (E_{n+1}, e_{n+1})$ such that $G|_{E_{n+1} \times \{s_0\}} = \text{identity}$ and $p_{n+1} \circ G$ is homotopic to $F \circ (p_{n+1} \times id)$. The above fact implies that if $(p_{n+1})_*(\beta) \in G_n(E_n, e_n)$ for $\beta \in \pi_n(E_{n+1}, e_{n+1})$, there exists $\gamma \in \operatorname{Ker}(p_{n+1})_*$ such that $\beta + \gamma \in G_n(E_{n+1}, e_{n+1})$. Thus $q'(\beta + \gamma) = q'\beta \in G_n(E_{n+1}, e_{n+1})$, that is $(p_{n+1})_*^{-1}(G_n(E_n, e_n))/G_n(E_{n+1}, e_{n+1}) \cap (p_{n+1})_*^{-1}(G_n(E_n, e_n)) \in \mathcal{C}_P$. Since $(p_{n+1})_*: \pi_n(E_{n+1}, e_{n+1}) \to \pi_n(E_n, e_n)$ is an epimorphism, we have $\pi_n(E_{n+1}, e_{n+1})/(p_{n+1})_*^{-1}(G_n(E_n, e_n)) \cong \pi_n(E_n, e_n)G_n(E_n, e_n) \in \mathcal{C}_P$. Therefore $\pi_n(E_{n+1}, e_{n+1})/G_n(E_{n+1}, e_{n+1}) \in \mathcal{C}_P$.

(iii) Suppose $m \geq n+2$. Since $n+1-m < 0$, $\mu F = 0 \in H^{n+1-m}(E_n; \pi)$. Noting that $(p_{n+1})_*: \pi_m(E_{n+1}, e_{n+1}) \to \pi_m(E_n, e_n)$ is an isomorphism, we can prove similarly as (i) and (ii) that $\pi_m(E_{n+1}, e_{n+1})/G_m(E_{n+1}, e_{n+1}) \in \mathcal{C}_P$.

Q.E.D.

Proof of Theorem 3. First assume that X is a mod $\boldsymbol{P}$ G-space. Let $\{X_i, g_i\}$ be a $\boldsymbol{P}$-sequence of X, where we may assume $X_i \simeq X$ for all i by Lemma 5. Choose an arbitrary integer $m(>1)$ and fix it. We will prove that $G_m(X_P, \bar{x}_0) = \pi_m(X_P, \bar{x}_0)$.

Let $\alpha \in \pi_m(X_P, \bar{x}_0)$ be an arbitrary element. Since $X_P = \bigcup_{i=0}^{\infty} X_i$, there exist an integer k and $\alpha_k \in \pi_m(X_k, x_k) \cong \pi_m(X, x_0)$ such that $(\bar{j}_k)_*(\alpha_k) = \alpha$, where $\bar{j}_k : X_k \to X_P$ is the obvious inclusion. Let q be the order of $\pi_m(X, x_0)/G_m(X, x_0)$, where $(q, p) = 1$ for all $p \in \boldsymbol{P}$. From the property [4; 1.1′, 2)′] of $\boldsymbol{P}$-sequences, it follows that there exist an integer $N(>k)$ and $\beta_N \in \pi_m(X_N, x_N) \cong \pi_m(X, x_0)$ such that $(g_N \circ \cdots \circ g_{k+1})_*(\alpha_k) = \alpha_N = q\beta_N$. Therefore $\alpha_N \in G_m(X_N, x_N)$. Let $Y = \bigcup_{i=N}^{\infty} X_i$, then Y may be considered as the localization of X_N at $\boldsymbol{P}$. By Lemma 4 we have $(j_{X_N})_*(\alpha_N) \in G_m(Y, y_0)$, where $j_{X_N} : X_N \to Y$ is the localization map and $y_0 = jX_N(x_N)$. It is clear that $\bar{j}_N : X_N \to X_P$ factors through Y, that is, there exists a homotopy equivalence $h : Y \to X_P$ such that $\bar{j}_N$ is homotopic to $h \circ j_{X_N}$. Thus $\alpha = (\bar{j}_N)_*(\alpha_N) = h_* \circ (j_{X_N})_*(\alpha_N)$. From Lemma 2 it follows $\alpha \in G_m(X_P, \bar{x}_0)$, since $(j_{X_N})_*(\alpha_N) \in G_m(Y, y_0)$ and h is a homotopy equivalence. Therefore $G_m(X_P, \bar{x}_0) = \pi_m(X_P, \bar{x}_0)$.

Conversely assume that X_P is a G-space. Let $\{p_n, E_n, (j_X)_n\}$ be a a Moore-Postnikov factorization of $j_X : X \to X_P$. Since a G-space is a mod $\boldsymbol{P}$ G-space, $E_1 = X_P$ is a mod $\boldsymbol{P}$ G-space. So using Lemma 8 we can prove by induction on n that $\pi_m(E_n, e_n)/G_m(E_n, e_n) \in \mathcal{C}_P$ for all m and n. From Lemma 6 it follows that X is a mod $\boldsymbol{P}$ G-space. Q.E.D.

References

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