103. A Note on H-Separable Extensions

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A ring extension A/B with common identity is called an *H-separable extension* if $A\otimes_B A$ is A-A-isomorphic to an A-A-direct summand of a finite direct sum of copies of A, and it is known that every H-separable extension is a separable extension (cf. for instance [4, p. 243]). As was shown in [5, Proposition 1.1], *if A is a separable R-algebra and a projective R-module then A is a finitely generated R-module.*

In this note, we shall prove an analogue of the above for H-separable extensions:

Proposition. *If A/B is an H-separable extension such that A_B is projective, then A_B is finitely generated.*

In virtue of the proposition, we see that in [1, Proposition 1.9, Corollary 1.6 and Theorem 1.3], [2, Theorem 4 and Corollary 2], [3, Theorem 1, Corollary 1 and Proposition 4] and [4, Proposition 2.1 and Theorem 2.2] the assumption that the extension considered is a finitely generated module over the ground ring is automatically satisfied. Especially, if A/B is an H-separable extension and B is Artinian simple then A_B is finitely generated free, which enables us to cut down the proof of [4, Theorem 1.5 (2)].

Now, our proposition is a direct consequence of the next easy lemma, since $A\otimes_B A_A$ is finitely generated for every H-separable extension A/B.

Lemma. *Let $\rho: B\to A$ be a ring monomorphism (sending 1 to 1), and M_B a projective module. Then, M_B is finitely generated if (and only if) $i_\rho(M)_A=M\otimes_B A_A$ is finitely generated.*

Proof. Let $\{u_\lambda; f_\lambda\}_{\lambda\in\Lambda}$ ($u_\lambda\in M, f_\lambda\in \mathrm{Hom}(M_B, B_B)$) be a projective coordinate system for M_B; i.e., $u=\sum_{\lambda\in\Lambda}u_\lambda f_\lambda(u)$ for every $u\in M$, $f_\lambda(u)$ being zero for almost all λ. Then, f_λ extends naturally to $f_\lambda^*\in\mathrm{Hom}(i_\rho(M)_A, A_A)$ and $\{u_\lambda\otimes 1; f_\lambda^*\}_{\lambda\in\Lambda}$ is a projective coordinate system for $i_\rho(M)_A$. Since $i_\rho(M)_A$ is finitely generated by hypothesis, we can find a finite subset K of Λ such that $\{u_\kappa\otimes 1\}_{\kappa\in K}$ is a generating system for $i_\rho(M)_A$. We consider here the set $I=\{\lambda\in\Lambda\,|\,f_\lambda(u_\kappa)\neq 0$ for some $\kappa\in K\}$, that is obviously a finite subset of Λ. If u is an arbitrary element of M then $u\otimes 1=\sum_{\kappa\in K}(u_\kappa\otimes 1)a_\kappa$ with some $a_\kappa\in A$. To be easily seen, we have then $\{\lambda\in\Lambda \mid f_\lambda^*(u\otimes 1)\neq 0\}\subseteq I$, so that $u\otimes 1=\sum_{\lambda\in\Lambda}(u_\lambda\otimes 1)f_\lambda^*(u\otimes 1)=\sum_{\iota\in I}(u_\iota\otimes 1)f_\iota^*(u\otimes 1)$, which implies $u=\sum_{\iota\in I}u_\iota f_\iota(u)$.

References

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