142. A Comment on the Galois Theory for Finite Factors

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1. As the case of simple rings, it is proved by Nakamura and Takeda ([6]–[8] and [9]) that a Galois theory holds true for finite factors under some conditions.

Throughout this paper, denote by $\mathfrak{H}$ a separable Hilbert space, by $\mathcal{A}$ a von Neumann algebra acting standardly on $\mathfrak{H}$, by G a countable discrete group of outer $*$-automorphisms of $\mathcal{A}$ and by $\mathcal{B}$ the fixed algebra of $\mathcal{A}$ under G, that is,

$$\mathcal{B}=\{A \in \mathcal{A};\, g(A)=A \text{ for all } g \in G\}.$$

Let $\mathcal{A}$ be a II_1-factor and G an outer automorphism group of $\mathcal{A}$. Then $\mathcal{A}$ is called a *Galois extension* of $\mathcal{B}$ with the *Galois group G* if $\mathcal{B}$ satisfies the condition:

(1) *The commutant $\mathcal{B}'$ of $\mathcal{B}$ is a II_1-factor.*

The fundamental theorem ([7, Theorem 2]) of the Galois theory for finite factors is the following:

Theorem A. *If $\mathcal{A}$ is a Galois extension of $\mathcal{B}$ with the Galois group G, then the lattices of all subgroups of G and of all intermediate subfactors $\mathcal{B}$ to $\mathcal{A}$ are dually isomorphic by the usual Galois correspondence.*

Furthermore the condition (1) is equivalent to the following condition:

(2) *G is finite*

([7, Theorem 3]).

The Galois theory for general von Neumann algebras is discussed by Haga and Takeda ([4]) or Henle ([5]).

In this paper, we shall show the following theorem as a comment of the Galois theory for II_1-factors.

Theorem 1. *Assume that $\mathcal{A}$ be a II_1-factor and G a finite group. Then the crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G is isomorphic to the tensor product $\mathcal{B}\otimes\mathcal{L}(l^2(G))$ of $\mathcal{B}$ and the algebra of all bounded linear operators on the Hilbert space $l^2(G)$.*

Recently, M. Choda in [1] introduced a notion of shift for automorphism groups. Relating to it, we shall characterize the shift for finite groups of automorphisms.

2. Now, we shall relate briefly as to the crossed product according to Haga and Takeda [4].

Let $G\otimes\mathfrak{H}$ be the Hilbert space of all formal sum $\sum_{g\in G} g\otimes\xi_g$ where the ξ_g are elements of $\mathfrak{H}$ and

$$\Big\|\sum_{g\in G} g\otimes\xi_g\Big\|_{G\otimes\mathfrak{H}}=\Big(\sum_{g\in G}\|\xi_g\|^2\Big)^{1/2}<+\infty.$$

Then $G\otimes\mathfrak{H}$ is the tensor product $l^2(G)\otimes\mathfrak{H}$. Define an operator $g\otimes A$ ($g\in G, A\in\mathcal{A}$) on $G\otimes\mathfrak{H}$ by

$$(g\otimes A)\Big(\sum_{h\in G} h\otimes\xi_h\Big)=\sum_{h\in G} hg^{-1}\otimes h(A)\xi_h.$$

Then direct computations show that

$$(g\otimes A)(h\otimes B)=gh\otimes h^{-1}(A)B,$$
$$(g\otimes A)^*=g^{-1}\otimes g(A^*)$$

and

$$(g\otimes I)(e\otimes A)(g\otimes I)^*=e\otimes g(A)$$

where e is the unit of G. The crossed product $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G is the von Neumann algebra generated by $\{g\otimes A\,; g\in G \text{ and } A\in\mathcal{A}\}$. Then we shall identify $e\otimes\mathcal{A}$ with $\mathcal{A}$ because $e\otimes\mathcal{A}=\{e\otimes A\,; A\in\mathcal{A}\}$ is isomorphic to $\mathcal{A}$.

3. Now, we shall show Theorem 1. Put

$$P=\frac{1}{n}\sum_{g\in G}(g\otimes I),$$

where n is the order of the finite group G. This projection P is used in the proof of Theorem 6.1.1 in [3]. By the definition of the crossed product, we have, for any $h\in G$ and any $A\in\mathcal{A}$,

$$\begin{aligned}P(h\otimes A)P&=\frac{1}{n}\Big(\sum_{g\in G} g\otimes I\Big)(h\otimes A)P\\&=\frac{1}{n}\Big(\sum_{g\in G} gh\otimes A\Big)P\\&=\frac{1}{n}\Big(\sum_{g\in G} g\otimes I\Big)(e\otimes A)P=P(e\otimes A)P,\end{aligned}$$

which implies that $P(G\otimes\mathcal{A})P=P\mathcal{A}P$. And also, we have $P\mathcal{A}P=P\mathcal{B}$. In fact, for any $A\in\mathcal{A}$,

$$\begin{aligned}P(e\otimes A)P&=\frac{1}{n^2}\sum_{g,h}(gh\otimes h^{-1}(A))=\frac{1}{n^2}\sum_{g}\Big(\sum_{h} h\otimes h^{-1}g(A)\Big)\\&=\frac{1}{n^2}\sum_{h}\Big(h\otimes\sum_{g} h^{-1}g(A)\Big)=\frac{1}{n}\sum_{h} h\otimes\Big(\frac{1}{n}\sum_{g} g(A)\Big)\\&=P\Big(e\otimes\frac{1}{n}\sum_{g} g(A)\Big).\end{aligned}$$

Therefore, we have

(3) $P(G\otimes\mathcal{A})P=P\mathcal{B}$.

On the other hand, P is a projection belonging to $\mathcal{B}'$ on $G\otimes\mathfrak{H}$ and $\mathcal{B}$ is a factor by Theorem A. Therefore $\mathcal{B}$ is isomorphic to the induced von Neumann algebra $\mathcal{B}_P$ of $\mathcal{B}$ induced by P ([2, p. 19]). Since $G\otimes\mathcal{A}$

is a II_1-factor and the trace of P is $1/n$, it follows that $G \otimes \mathcal{A}$ is isomorphic to

$$(G \otimes \mathcal{A})_P \otimes \mathcal{L}(\mathfrak{H}_n),$$

where $\mathfrak{H}_n$ is the n-dimensional Hilbert space.

Therefore, $G \otimes \mathcal{A}$ is isomorphic to

$$\mathcal{B}_P \otimes \mathcal{L}(l^2(G))$$

by (3), and so, to $\mathcal{B} \otimes \mathcal{L}(l^2(G))$.

The proof of Theorem 1 is a modification of the proof of Theorem 6.1.1 in Golodets [3]. This Theorem 1 implies Theorem 6.1.1 of [3]:

Let $\mathcal{A}$ be a II_1-factor and G a finite group of outer automorphisms of $\mathcal{A}$, then the fixed algebra $\mathcal{B}$ of $\mathcal{A}$ under G is hyperfinite if and only if $G \otimes \mathcal{A}$ is hyperfinite.

4. Very recently, M. Choda has introduced the notion of shift for automorphism groups, which is very interesting.

If there exists a projection P in $\mathcal{A}$ such that

(4) $g(P)P=0$ for each $g(\neq e) \in G$

and

(5) $\sum_{g \in G} g(P)=1,$

then G is called a *shift* on $\mathcal{A}$. Especially, if G is finite and abelian, then the shift is characterized as the following:

Theorem 2. *A finite abelian group G is shift on $\mathcal{A}$ if and only if there exists a unitary representation U_γ of the dual $\hat{G}$ in $\mathcal{A}$ such that*

(6) $g(U_\gamma)=\langle g, \gamma \rangle U_\gamma$ *for every* $g \in G$,

where $\langle g, \gamma \rangle$ is the value of γ at g.

Proof. If G is a shift, then there exists a projection P of $\mathcal{A}$ satisfying (4) and (5). For every γ of $\hat{G}$, put

$$U_\gamma = \sum_{g \in G} \overline{\langle g, \gamma \rangle} g(P).$$

Then U_γ is a unitary of $\mathcal{A}$ with the property (6).

Conversely, assume that there exists a unitary representation U_γ of $\hat{G}$ in $\mathcal{A}$ with the property (6). Put

$$P = \frac{1}{n} \sum_{\gamma \in \hat{G}} U_\gamma,$$

where n is the order of G. Then P is a projection of $\mathcal{A}$ satisfying (4) and (5).

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