140. **Double Centralizers of Torsionless Modules**

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In this note, we make the assumption that a ring has an identity element and modules are unital. For a left $R$-module $_R M$ where $R$ is a ring, $D = \text{End}_R (_R M)$ is an $R$-endomorphism ring of $_R M$ operating on the side opposite to the scalars. Then $_R M$ is considered as an $(R, D)$-bimodule. A $D$-endomorphism ring $Q = \text{End}_D (M_D)$ of $M_D$ is called a double centralizer of $_R M$.

**Definition.** Let $_R M$ and $_U M$ be left $R$-modules, $_R M$ is said to be $U$-torsionless in case for each non-zero element $m$ of $_R M$, there exists an $R$-homomorphism $\phi$ of $_R M$ into $_U M$ such that $(m)\phi \neq 0$.

We say that a left $R$-module $_R M$ is torsionless if $_R M$ is $R$-torsionless and $_N N$ is faithful if $_R N$ is $N$-torsionless. Let $Q$ be a double centralizer of a faithful left $R$-module $_R M$, then there exists a canonical ring monomorphism of $R$ into $Q$, written as $R \subseteq Q$. A faithful left $R$-module $_R M$ is said to have the double centralizer property if $R = Q$, where $Q$ is a double centralizer of $_R M$.

**Definition.** A ring $R$ is left $QF$-1 if every faithful left $R$-module has the double centralizer property.

$QF$-1 rings were first described by R. M. Thrall (1948 [4]) and have been examined by many authors. It was proved that the double centralizer of a faithful torsionless left $R$-module is a rational extension of $R_R$. Furthermore the double centralizer of a dominant left $R$-module is a maximal right quotient ring of $R$ (see T. Kato [1] and H. Tachikawa [3]). In the section 1, the next theorem is proved.

**Theorem.** Let $R$ be a ring with minimum condition and $U$ be the intersection of all left faithful two-sided ideals of $R$. Then $U$ is also a left faithful two-sided ideal of $R$ and the double centralizer of $_R U$ is a maximal right quotient ring of $R$.

In the section 2, we shall prove that for a given faithful left $R$-module $_R M$, $_R M$ has the double centralizer property if and only if $_K Ke$ has the double centralizer property, where

$$K = \left( \begin{array}{cc} R & M \\ \text{Hom}_R (_R M, _R R) & \text{End}_R (_R M) \end{array} \right) \quad \text{and} \quad e = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \in K.$$

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*1) Dedicated to professor Kiiti Morita on his 60th birthday.
1. We shall first prove the next theorem which is similar to K. Morita’s result [2, Theorem 1.1.].

**Theorem 1.** Let \( rM \) and \( rU \) be left \( R \)-modules. If the following conditions are satisfied:

(1) There exists the following \( R \)-exact sequence:

\[
\bigoplus_{rM} \to rU \to 0,
\]

(2) If \( \sum m_i \phi_i = 0 \), \( m_i \in rM \), \( \phi_i \in \text{Hom}_R (rM, rU) \), then \( \sum (qm_i) \phi_i = 0 \)

for any \( q \in Q \),

(3) For each non-zero element \( q \) of \( Q \), there exist \( m \in rM \) and \( \phi \in \text{Hom}_R (rM, rU) \) such that \( (qm) \phi \neq 0 \),

then we have \( Q \subset Q \) where \( Q \) and \( \overline{Q} \) are double centralizers of \( rM \) and \( rU \) respectively.

**Proof.** For any \( q \in Q \), we define \( \bar{q} \) as \( \bar{q}(\sum m_i \phi_i) = \sum (qm_i) \phi_i \). Then the mapping \( q \to \bar{q} \) is well-defined by (2). An element \( \bar{q} \) is contained in \( Q \) since

\[
q((m)d) = q(md) = (q(m)id) = (q(\sum m_i \phi_i))d
\]

for any \( d \in \text{End}_R (rU) \). And this mapping \( q \to \bar{q} \) is a ring monomorphism of \( Q \) into \( Q \) by (3).

**Lemma 2.** If \( rU \) is \( rM \)-torsionless, then the condition (2) of Theorem 1 is satisfied.

**Proof (c.f. T. Kato [1]).** If \( \sum (qm_i) \phi_i \neq 0 \), \( q \in Q \), \( m_i \in rM \), \( \phi_i \in \text{Hom}_R (rM, rU) \), then there exists \( d \in \text{Hom}_R (rU, rM) \) such that \( (\sum (qm_i) \phi_i) d \neq 0 \) since \( rU \) is \( rM \)-torsionless. And

\[
(\sum (qm_i) \phi_i) d = \sum (qm_i) \phi_i d = \sum q(m_i \phi_i d)
\]

by \( q \in Q = \text{Hom}_R (rM, rU) \). Further

\[
\sum q(m_i \phi_i d) = \sum (m_i \phi_i d) = q(\sum m_i \phi_i d) \neq 0.
\]

Then we have \( \sum m_i \phi_i \neq 0 \).

Since the condition (3) of Theorem 1 is satisfied if \( rM \) is \( rU \)-torsionless, we have the following.

**Lemma 3.** Let \( rM \) and \( rU \) be left \( R \)-modules. If the following conditions are satisfied:

(1) There exists the following \( R \)-exact sequence:

\[
\bigoplus_{rM} \to rU \to 0,
\]

(2) \( rU \) is \( rM \)-torsionless,

(3) \( rM \) is \( rU \)-torsionless,

then we have \( Q \subset Q \) where \( Q \) and \( \overline{Q} \) are double centralizers of \( rM \) and \( rU \) respectively.

**Lemma 4.** Let \( A \) and \( B \) be left faithful two-sided ideals of a ring \( R \). Then \( A \cap B \) is also a left faithful two-sided ideal of \( R \).

**Proof.** Clearly \( AB = \{ \sum a_i b_i | a_i \in A, b_i \in B \} \) is a two-sided ideal contained in a two-sided ideal \( A \cap B \). For each non-zero element \( r \) of \( R \), there exists \( a \in A \) such that \( ra \neq 0 \) since \( A \) is left faithful. Similarly
for \( ra \neq 0 \), there exists \( b \in B \) such that \((ra)b = 0\) since \( B \) is left faithful. Thus, \( r(ab) \neq 0 \), \( ab \in AB \), \( AB \) is left faithful. Hence \( A \cap B \) is also left faithful.

**Definition.** For a left \( R \)-module \( _RM \), the sum of all \( R \)-homomorphic images of \( _RM \) into \( _RR \) is called a trace ideal of \( _RM \), written as \( \text{Tr} \( _RM \) \).

**Theorem 5.** Let \( R \) be a ring with minimum condition and \( U \) be the intersection of all left faithful two-sided ideals of \( R \). Then \( U \) is also a left faithful two-sided ideal of \( R \) and the double centralizer of \( _RU \) is a maximal right quotient ring of \( R \).

**Proof.** If \( R \) is a ring with minimum condition, then \( U \) is a left faithful two-sided ideal of \( R \) because of Lemma 4. For any faithful torsionless left \( R \)-module \( _RM \), let \( \text{Tr} \( _RM \) \) be a trace ideal of \( _RM \) and \( Q, Q' \) be double centralizers of \( _RM, _R\text{Tr} \( _RM \) \) respectively. By Lemma 3, we have \( Q \subseteq Q' \). Since \( \text{Tr} \( _RM \) U \) is also a left faithful two-sided ideal contained in \( U \), then \( \text{Tr} \( _RM \) U = U \). In this case, \( Q' \) is contained in the double centralizer \( Q \) of \( _RU \) by Lemma 3. Thus we have \( Q \subseteq Q' \).

This ring \( Q \) is a maximal right quotient ring of \( R \) since \( R \) has a dominant left \( R \)-module (see T. Kato [1]).

**Theorem 6.** Let \( R \) be a left cogenerator ring. Then the following statements are equivalent:

1. \( R \) is a left QF-1 ring.
2. Every faithful left ideal of \( R \) has the double centralizer property.
3. Every left faithful two-sided ideal of \( R \) has the double centralizer property.
4. Every left faithful trace ideal of \( R \) has the double centralizer property.

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) is clear. (4) \( \Rightarrow \) (1). For any faithful left \( R \)-module \( _RM \), let \( \text{Tr} \( _RM \) \) be a trace ideal of \( _RM \) and \( Q, Q' \) be double centralizers of \( _RM, _R\text{Tr} \( _RM \) \) respectively. By Lemma 3, we have \( Q \subseteq Q' \). This implies \( Q \subseteq Q' \).

2. In this section, let \( _RM \) be a faithful left \( R \)-module, \( D = \text{End}_R \( _RM \) \) and \( Q = \text{End}_D \( M_D \) \). It is easily shown that the canonical mapping

\[
\eta : \text{Hom}_R \( _RM, _RR \) \rightarrow \text{Hom}_D \( M_D, D_D \)
\]

is a \((D, R)\)-monomorphism and the canonical mapping

\[
\rho : \text{Hom}_D \( M_D, D_D \) \rightarrow \text{Hom}_Q \( qM, qQ \)
\]

is a \((D, Q)\)-isomorphism. We define a ring \( K \) as

\[
K = \left\{ \begin{pmatrix} r & \phi & m \\ \phi & d \\ d \end{pmatrix} \mid r \in R, m \in M, \phi \in \text{Hom}_R \( _RM, _RR \), d \in D \right\}.
\]
In this ring $K$, for $m \in M$ and $\phi \in \text{Hom}_R (\_M, \_R)$, let $m\phi \in R$ be as usual but $\phi m$ means $(\eta \phi)m \in D$.

**Lemma 7.** Let $M$, $D$, $Q$ and $K$ be as above. If $M$ is faithful, then $K\text{Ke}$ is faithful, where

$$
eq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in K.$$

**Lemma 8.** Let $M$, $D$, $Q$, $K$ and $e \in K$ be as above. Then the double centralizer of $K\text{Ke}$ is a ring

$$\begin{pmatrix} Q & M \\ \text{Hom}_D (M_D, D_D) & D \end{pmatrix}.$$

Finally we describe our main theorem which is thought to be useful in solving later problems.

**Theorem 9.** Let $M$, $D$, $Q$, $K$ and $e \in K$ be as above. Then $M$ has the double centralizer property if and only if $K\text{Ke}$ has the double centralizer property.

**References**