138. *On the Strict Union of Ranked Metric Spaces*

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The main purpose of this paper is to give the definition of the "*union*" of ranked spaces and to show that

(i) properties relating to convergence in the strict union E of ranked metric spaces $E_\alpha(\alpha \in A)$ are reduced to those in E_α (Theorem 1),

(ii) if $E_\alpha(\alpha \in A)$ is directed under set theoretic inclusion, a completion of E results from the completion of each E_α (Theorem 2).

Throughout of this paper, a "*ranked space*" means a ranked space of indicator ω_0(ω_0 is the first nonfinite ordinal) defined as follows:

1. Definition of the ranked space (of indicator ω_0). Let us consider a non-empty set E in which each point p has a non-empty family consisting of subsets of E, denoted by $V(p), U(p), \cdots$ and called *preneighborhoods* (sometimes called neighborhoods) *of* p or *p-preneighborhoods*, such that

(A) $p \in V(p)$.

We denote the family of all p-preneighborhoods by $\mathcal{V}(p)$, and put $\mathcal{V}=\{V(p); V(p) \in \mathcal{V}(p), p \in E\}$. Moreover, we assume that for each $n \in N$ ($N=\{0,1,2,\cdots\}$), we have a family $\mathcal{V}_n$ of preneighborhoods, called *preneighborhoods of rank n*, which satisfies the following axiom (a):

(a) For each preneighborhood $V(p)$ of p and for each number $n \in N$, there exist an m and a $U(p)$ which satisfy at the same time $U(p) \subset V(p)$, $U(p) \in \mathcal{V}_m$ and $n \leq m$. A *ranked space* (*of indicator* ω_0) is a non-empty set E endowed with these families $\mathcal{V}, \mathcal{V}_n (n \in N)$, which is written $(E, \mathcal{V}, \mathcal{V}_n)$ (briefly, $(E, \mathcal{V})$ or E). p-preneighborhoods of rank n are written $V(p, n)$.

We recall a few definitions concerning ranked spaces. $\{V(p_i, n_i); i=0,1,2,\cdots\}$ (briefly, $\{V(p_i, n_i)\}$) is called *fundamental* if $V(p_0, n_0) \supset V(p_1, n_1) \supset \cdots$ and for every i, there is $i_0 \geq i$ such that $p_{2i_0}=p_{2i_0+1}$ and $n_{2i_0}<n_{2i_0+1}$. E is called *complete* if for every fundamental sequence $\{V(p_i, n_i)\}$, $\cap V(p_i, n_i) \neq \phi$. $\{p_i\}$ is said to *r-converge to* p if there is a fundamental sequence $\{V(p, n_i)\}$ such that $p_i \in V(p, n_i)$ for every i. In this case, we write $p \in \{r\text{-}\lim p_n\}$.

2. Definition of the union of ranked spaces. Let $(E_\alpha, \mathcal{V}^\alpha, \mathcal{V}^\alpha_n)$ $(\alpha \in A)$ be a family of the ranked spaces and let $E=\cup E_\alpha$. For each $p \in E$, we consider p-preneighborhoods in every E_α such that $p \in E_\alpha$, and denote the family of all such p-preneighborhoods by $\mathcal{V}(p)$. Then, as it is easily seen, $(E, \mathcal{V}, \mathcal{V}_n)$ is a ranked space if we put $\mathcal{V} = \{V(p)\,;\, V(p) \in \mathcal{V}(p), p \in E\}$, $\mathcal{V}_n = \{V(p)\,;\, V(p) \in \mathcal{V}_n^\alpha (\alpha \in A)\}$ for each $n \in N$. We call this space the *ranked union space*, or briefly *union*, of $(E_\alpha, \mathcal{V}^\alpha, \mathcal{V}_n^\alpha)$ $(\alpha \in A)$. In this space, $V(p, n) \in \mathcal{V}_n$ are written $V(p, \alpha, n)$ if $V(p, n) \in \mathcal{V}_n^\alpha$.

The following assertion is immediate from the definition.

Proposition 1. *Let $E, E_\alpha (\alpha \in A)$ be as in definition above, then each fundamental sequence of preneighborhoods in $E_\alpha (\alpha \in A)$ is a fundamental sequence in E.*

3. Definition of the r-Cauchy sequence. Let E be an arbitrary ranked space. A fundamental sequence $\{V(p_i, n_i)\}$ in E is called *canonical* if $p_{2i} = p_{2i+1}$ for every i and $n_0 < n_1 < \cdots < n_i < \cdots$. A sequence $\{p_n\,;\, n = 0, 1, 2, \cdots\}$ such that every p_n (or except the first finite p_n) belongs to E, is called *r-Cauchy sequence* if there is a canonical fundamental sequence $\{V(q_i, m_i)\}$ in E such that

(i) $\{q_{2i}\}$ is a subsequence of $\{p_n\}$ written $\{p_{n_i}\}$,

(ii) $V(q_{2i+1}, m_{2i+1}) \ni p_{n_i+1}, p_{n_i+2}, \cdots$ for every i.

4. Definition of the ranked metric space. Let (E, d) be a metric space with a distance function d and let $\bar{R}^+$ be the set of extended positive real numbers. We define the family $\mathcal{V}(p)$ of preneighborhoods of $p \in E$ as follows:

$\mathcal{V}(p) = \{V_\lambda(p)\,;\, \lambda \in \bar{R}^+\}$, where $V_\lambda(p) = \{q\,;\, d(p, q) < \lambda\}$. We put $\mathcal{V}(d) = \{V_\lambda(p)\,;\, V_\lambda(p) \in \mathcal{V}(p),\, \lambda \in \bar{R}^+,\, p \in E\}$, and put for each $n \in N$, $\mathcal{V}_n(d) = \{V_{1/2^n}(p)\,;\, p \in E\}$ if $n \neq 0$ and $\mathcal{V}_n(d) = \{V_{+\infty}(p)\,;\, p \in E\}$ if $n = 0$. Then, $(E, \mathcal{V}(d), \mathcal{V}_n(d))$ becomes a ranked space, called a *ranked metric space.*

The notions relating to convergence in (E, d) are translated into the corresponding notions in $(E, \mathcal{V}(d))$ as follows:

Proposition 2. (i) *$\{r\text{-}\lim p_n\} \ni p$ in $(E, \mathcal{V}(d))$ iff $\lim p_n = p$ in (E, d)*; (ii) *$\{p_n\}$ is an r-Cauchy sequence in $(E, \mathcal{V}(d))$ iff $\{p_n\}$ is a Cauchy sequence in (E, d)*; (iii) *$(E, \mathcal{V}(d))$ is complete iff (E, d) is complete.*

(i) and (ii) are already proved in [1].

5. Definition of the r-continuity. Let E, F be ranked spaces and f be a map of E into F. Then, f is called *r-continuous at $p \in E$* if for every fundamental sequence $\{V(p, n_i)\}$ in E there is a fundamental sequence $\{V(f(p), m_i)\}$ in F such that $f(V(p, n_i)) \subset V(f(p), m_i)$ for every i. f is called *r-continuous on E into F* if f is r-continuous at each $p \in E$. (In general, $f(A)$ denotes $\{f(p)\,;\, p \in A\}$.)

Proposition 3. *In the definition above, if F is a ranked metric space, f is r-continuous at p iff for every $\{p_n\}$ in E such that $\{r\text{-}\lim p_n\} \ni p$, we have $\{r\text{-}\lim f(p_n)\} \ni f(p)$. (Prof. K. Kunugi has used in* [2] *this property to define r-continuity.)*

6. Definition of the strict union of ranked metric spaces. The union of ranked metric spaces $(E_\alpha, \mathcal{V}(d_\alpha), \mathcal{V}_n(d_\alpha))$ $(\alpha \in A)$ is called the *ranked strict union space*, or briefly *strict union, of ranked metric spaces* if it satisfies the following condition $(\dagger_1)$:

$(\dagger_1)$ $\quad d_\alpha(p,q)=d_{\alpha'}(p,q)$ $\quad$ for every pair $p, q \in E_\alpha \cap E_{\alpha'}$.

Lemma 1. *Let E be the strict union of ranked metric spaces $(E_\alpha, \mathcal{V}(d_\alpha))$ $(\alpha \in A)$. Then*

(i) *if $E_\alpha \supset V(p, \alpha', n)$, we have $V(p, \alpha, n) \supset V(p, \alpha', n)$,*

(ii) *if $V(p, \alpha, n) \supset V(p, \alpha, m) \ni q$ and $n<m$, we have $V(p, \alpha, n) \supset V(q, \alpha, m)$,*

(iii) *if $\{V(p_i, \alpha_i, n_i)\}$ is a canonical fundamental sequence in E, $\{V(p_{i+1}, \alpha_0, n_i)\}$ is a fundamental sequence in E_{α_0} such that $V(p_{i+1}, \alpha_0, n_i) \supset V(p_{i+1}, \alpha_{i+1}, n_{i+1})$ for every $i \in N$.*

Proof. (i) and (ii) follow from $(\dagger_1)$ and the property of distance function, respectively. (iii) is proved by using (i), (ii) and the property of distance function, repeatedly.

Theorem 1. *Let E be as in Lemma 1. Then*

(i) *$\{r\text{-}\lim p_n\} \ni p$ in E iff $\{r\text{-}\lim p_n\} \ni p$ in some E_α,*

(ii) *$\{p_n\}$ is an r-Cauchy sequence in E iff $\{p_n\}$ is that in some E_α,*

(iii) *E is complete iff every E_α $(\alpha \in A)$ is complete.*

Furthermore, if f is a map of E into a ranked space F, f_α is the restriction of f to E_α and $p \in E$, then

(iv) *f is r-continuous at p in E iff, for every $\alpha \in A$ such that $p \in E_\alpha$, f_α is r-continuous at p in (E_α, d_α).*

(v) *f is r-continuous on E into F iff, for every $\alpha \in A$, f_α is r-continuous on E_α into F.*

Proof. Sufficiency in (i), (ii) and necessity in (iii), (iv) are immediate from Proposition 1. Necessity in (i) and sufficiency in (iv) are proved by using Lemma 1, (i), and necessity in (ii) is proved by using Lemma 1, (iii). (v) follows from (iv). Next, we shall prove sufficiency in (iii). Without loss of generality, it is sufficient to prove that for every canonical fundamental sequence $\{V(p_i, \alpha_i, n_i)\}$, $\cap V(p_i, \alpha_i, n_i) \neq \phi$. Since then $\{p_i\}$ is a Cauchy sequence in E_{α_0}, there is a $p \in E_{\alpha_0}$ such that $\lim p_i = p$ in E_{α_0} by the assumption. Then we can prove that $p \in \cap V(p_i, \alpha_i, n_i)$. Because, for every $i_0 \in N$, if we consider the canonical fundamental sequence $\{V(p_{2i_0+j}, \alpha_{2i_0+j}, n_{2i_0+j})\}$ in E, $\{V(p_{2i_0+j+1}, \alpha_{2i_0}, n_{2i_0+j})\}$ is a fundamental sequence in $E_{\alpha_{2i_0}}$ by Lemma 1, (ii). Therefore by the assumption, there is a $q \in V(p_{2i_0}, \alpha_{2i_0}, n_{2i_0})$ such that $\lim_j p_{2i_0+j+1} = q$ in $E_{\alpha_{2i_0}}$. Moreover, we have $p_{2i_0+j+1} \in E_{\alpha_0}$, $q \in E_{\alpha_0}$. Hence $\lim_j p_{2i_0+j+1} = q$ in E_{α_0}, and so $p=q$. Therefore $p \in V(p_{2i_0}, \alpha_{2i_0}, n_{2i_0})$ for every i_0. Thus $p \in \cap V(p_i, \alpha_i, n_i)$.

Corollary. *If E is as in Lemma 1, E is complete iff every r-*

Cauchy sequence r-converges.

7. Definition of the isomorphism. Ranked spaces $(E, \mathcal{V}, \mathcal{V}_n)$ and $(F, \mathcal{W}, \mathcal{W}_n)$ are called *isomorphic* if there is a bijection φ of E onto F such that for every $V(p) \in \mathcal{V}_n(p)$, $\varphi(V(p)) \in \mathcal{W}_n(\varphi(p))$ and for every $W(p) \in \mathcal{W}_n(p)$, $\varphi^{-1}(W(p)) \in \mathcal{V}_n(\varphi^{-1}(p))$; φ is called an *isomorphism* of E onto F.

8. Definition of the completion of ranked space. We say that, to an incomplete ranked space $(E, \mathcal{V}, \mathcal{V}_n)$, a ranked space $(F, \mathcal{W}, \mathcal{W}_n)$ is a *completion* of $(E, \mathcal{V}, \mathcal{V}_n)$ if it satisfies the following three conditions:

(i) $(F, \mathcal{W}, \mathcal{W}_n)$ is complete.

(ii) There is a ranked subspace F_0 of F which is isomorphic to E. (For the definition of the ranked subspace, see [3].)

(iii) $cl_r(F_0)$ $(=\{p\,;\, p \in \{r\text{-lim}\, p_n\}, p_n \in F_0\}) = F$.

Proposition 4. *If a metric space (F, d_1) is the completion of a metric space (E, d), the ranked metric space $(F, \mathcal{V}(d_1))$ is a completion of the ranked metric space $(E, \mathcal{V}(d))$.*

9. Completion of the strict union of ranked metric spaces. Throughout the following, let (E_α, d_α) $(\alpha \in A)$ be a family which satisfies $(\dagger_1)$ and the following condition $(\dagger_2)$:

$(\dagger_2)$ A is directed under "$\alpha \leq \alpha'$ if $E_\alpha \subset E_{\alpha'}$".

In this case, there is a distance function d of $E (= \cup E_\alpha)$ such that each d_α is the restriction of d to E_α.

To define a completion of the union of such a family, we first consider the completion $(\bar{E}_\alpha, \bar{d}_\alpha)$ of each (E_α, d_α) with an isometry φ_α of E_α into a dense subspace of $\bar{E}_\alpha$, and put $\bar{E} = \cup \bar{E}_\alpha$. Denote points of $\bar{E}$ by $\bar{p}, \bar{q}, \cdots$. Then

Lemma 2. *For every $E_\alpha, E_{\alpha'}$ such that $E_\alpha \subset E_{\alpha'}$, there is a one-to-one map $\varphi_{\alpha\alpha'}$ of $\bar{E}_\alpha$ into $\bar{E}_{\alpha'}$ such that*

(i) $\varphi_{\alpha\alpha} = 1_{\bar{E}_\alpha}$ *for each* $\alpha \in A$,

(ii) *if* $E_\alpha \subset E_{\alpha'}$,

(1°) $\varphi_{\alpha\alpha'}(\varphi_\alpha(p)) = \varphi_{\alpha'}(p)$ *for every* $p \in E_\alpha$,

(2°) $\bar{d}_\alpha(\bar{p}, \bar{q}) = \bar{d}_{\alpha'}(\varphi_{\alpha\alpha'}(\bar{p}), \varphi_{\alpha\alpha'}(\bar{q}))$ *for every pair* $\bar{p}, \bar{q} \in \bar{E}_\alpha$,

(iii) *if* $E_\alpha \subset E_{\alpha'} \subset E_{\alpha''}$, $\varphi_{\alpha\alpha''}(\bar{p}) = \varphi_{\alpha'\alpha''}(\varphi_{\alpha\alpha'}(\bar{p}))$ *for every* $\bar{p} \in \bar{E}_\alpha$.

Lemma 3. *For arbitrary points $\bar{p}, \bar{q}$ of $\bar{E}$ such that $\bar{p} \in \bar{E}_\alpha, \bar{q} \in \bar{E}_{\alpha'}$, we define $\bar{p} \sim \bar{q}$ if $\varphi_{\alpha\alpha''}(\bar{p}) = \varphi_{\alpha'\alpha''}(\bar{q})$ for an $\alpha'' \in A$ for which $E_{\alpha''} \supset E_\alpha \cup E_{\alpha'}$. Then, "$\sim$" is an equivalence relation between points of $\bar{E}$.*

We denote equivalence classes in $\bar{E}$ with respect to "$\sim$" defined above by $\hat{p}, \hat{q}, \cdots$. For each $\alpha \in A$, we denote the set of all those $\hat{p}$ containing $\bar{p}$ such that $\bar{p} \in \bar{E}_\alpha$ by $\hat{E}_\alpha$, and put $\hat{E} = \cup \hat{E}_\alpha$. Then

Lemma 4. (i) *If we let ψ be the map of $\bar{E}$ onto $\hat{E}$ such that $\psi(\bar{p}) = \hat{p}$, where $\hat{p} \ni \bar{p}$, the restriction ψ_α of ψ to $\bar{E}_\alpha$ is a bijection of $\bar{E}_\alpha$ onto $\hat{E}_\alpha$.*

(ii) *If we put* $\hat{d}_\alpha(\hat{p}, \hat{q}) = \bar{d}_\alpha(\psi_\alpha^{-1}(\hat{p}), \psi_\alpha^{-1}(\hat{q}))$ *for each pair* $\hat{p}, \hat{q} \in \hat{E}_\alpha$, $\hat{d}_\alpha$ *is a distance function in* $\hat{E}_\alpha$.

(iii) *If* $p \in E_\alpha \cap E_{\alpha'}$, $\psi(\varphi_\alpha(p)) = \psi(\varphi_{\alpha'}(p))$.

(iv) *If* $E_\alpha \subset E_{\alpha'}$, $\hat{E}_\alpha \subset \hat{E}_{\alpha'}$ *and* $\hat{d}_\alpha(\hat{p}, \hat{q}) = \hat{d}_{\alpha'}(\hat{p}, \hat{q})$ *for every pair* $\hat{p}, \hat{q} \in \hat{E}_\alpha$.

(v) $(\hat{E}_\alpha, \hat{d}_\alpha)$ *is the completion of* (E_α, d_α) *for every* $\alpha \in A$.

(vi) $(\hat{E}_\alpha, \hat{d}_\alpha)$ $(\alpha \in A)$ *satisfies* $(\dagger_1)$ *and* $(\dagger_2)$.

We now get the following theorem by using Theorem 1 and Lemma 4.

Theorem 2. *The strict union of* $(\hat{E}_\alpha, \mathcal{V}(\hat{d}_\alpha))$ $(\alpha \in A)$ *is a completion of the strict union of* $(E_\alpha, \mathcal{V}(d_\alpha))$ $(\alpha \in A)$.

Remark. For a family (E_α, d_α) $(\alpha \in A)$ satisfying $(\dagger_1)$, and $E = \bigcup E_\alpha$, if $(E, \mathcal{V})$ is the strict union of $(E_\alpha, \mathcal{V}(d_\alpha))$ $(\alpha \in A)$ and $(E, \mathcal{T})$ is the topological space with the inductive topology $\mathcal{T}$ with respect to canonical imbeddings $E_\alpha \to E$, then

(i) the notion of r-continuity of $(E, \mathcal{V})$ into $(F, \mathcal{V}(d))$ coincides with that of continuity on $(E, \mathcal{T})$ into (F, d).

In particular, if the family is a sequence $E_n (n \in N)$ of (metrizable) locally convex topological vector spaces such that $E_0 \subset E_1 \subset \cdots$ and E_n is closed in (E_{n+1}, d_{n+1}), then

(ii) the notions of r-convergence, r-Cauchy sequence and completeness in $(E, \mathcal{V})$ coincide with those of convergence of sequence, Cauchy sequence and completeness in $(E, \mathcal{T})$, respectively.

References

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