136. Projective Modules and 3-fold Torsion Theories

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(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1974)

Let R be a ring with identity and R-mod the category of unital left R-modules. A 3-fold torsion theory for R-mod is a triple $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ of classes of left R-modules such that both $(\mathfrak{T}_1, \mathfrak{T}_2)$ and $(\mathfrak{T}_2, \mathfrak{T}_3)$ are torsion theories for R-mod in the sense of Dickson [2]. A class $\mathfrak{T}_2$ for which there exist classes $\mathfrak{T}_1$ and $\mathfrak{T}_3$ such that $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ is a 3-fold torsion theory for R-mod will be called a TTF-class following Jans [3]. In this case, $\mathfrak{T}_1$-torsion submodule $t_1(M)$ and $\mathfrak{T}_2$-torsion submodule $t_2(M)$ coincide with $t_1(R)\cdot M$ and $r_M(t_1(R))$ respectively for any left R-module M (cf. [4, Lemma 2.1]), where $r_M(*)$ denotes the right annihilator of $*$ in M.

An idempotent two-sided ideal I of R determines three classes of left R-modules

$$\mathfrak{C}_I = \{{}_RM \mid IM = M\},$$
$$\mathfrak{T}_I = \{{}_RM \mid IM = 0\}$$

and

$$\mathfrak{F}_I = \{{}_RM \mid r_M(I) = 0\},$$

and $(\mathfrak{C}_I, \mathfrak{T}_I, \mathfrak{F}_I)$ is then a 3-fold torsion theory for R-mod. In this case, the $\mathfrak{C}_I$-torsion submodule and $\mathfrak{T}_I$-torsion submodule of a left R-module M coincide with IM and $r_M(I)$ respectively.

Recently, in his paper [1], Azumaya has proved that, among other things, for a 3-fold torsion theory $(\mathfrak{C}_I, \mathfrak{F}_I, \mathfrak{C}_I)$ determined by the trace ideal I of a projective R-module P, a necessary and sufficient condition for $\mathfrak{C}_I$ to be a TTF-class is that ${}_{R/l_R(I)}P$ is a generator for $R/l_R(I)$-mod. In this note we shall give a similar condition for $\mathfrak{F}_I$ to be a TTF-class and look at the result due to Azumaya again from our point of view. Throughout this note, R-modules will mean left R-modules and $l(*)$ $(r(*))$ will denote the left (right) annihilator for $*$ in R.

We shall begin with a lemma which is in need of later discussions.

Lemma 1. *Let I be a left ideal and K a right ideal in R. Then the following conditions are equivalent:*

(1) $I+K=R$.

(2) *For any R-module M, $IM=0$ implies that $KM=M$.*

If this is the case and if we assume moreover that $IK=0$, then

(3) *both I and K are idempotent two-sided ideals of R and $I=l(K)$ and $K=r(I)$, and*

(4) $\mathfrak{T}_I=\mathfrak{C}_K$.

In case I is an idempotent two-sided ideal in R and K is the trace ideal of a projective R-module P, then (4) *is equivalent to*

(5) ${}_{R/I}P$ *is a generator for* R/I-mod.

The proof is not so difficult except for the last part. (3) of this lemma is due to [1, Lemma 1]. As is easily seen, $\mathfrak{T}_I=\mathfrak{C}_K$ means that $IK=0$ (or, equivalently, $IP=0$) and $\mathfrak{T}_I\subset\mathfrak{C}_K$. This also means that P is an R/I-module and is a generator for R/I-mod, since $\mathfrak{C}_K$ consists of those R-modules which are epimorphic images of direct sums of copies of P.

We shall say that a 3-fold torsion theory $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ for R-mod has length 2 if $\mathfrak{T}_1=\mathfrak{T}_3$.

The first halves of the following propositions may be seen as slightly different versions of [1, Theorem 3].

Proposition 2. *Let* $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ *be a 3-fold torsion theory for R-mod. Then* $\mathfrak{T}_3$ *is a* TTF-*class if and only if*

$$t_1(R)+r(t_1(R))=R.$$

Moreover, if this is the case, $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ *has length* 2 *if and only if* $r(t_1(R))\cdot t_1(R)=0$.

Proof. Suppose that $\mathfrak{T}_3$ is a TTF-class. Then there exists a class $\mathfrak{T}$ of R-modules such that $(\mathfrak{T}_2, \mathfrak{T}_3, \mathfrak{T})$ is also a 3-fold torsion theory for R-mod and so by [4, Lemma 2.1] $\mathfrak{T}_2=\mathfrak{C}_{r(t_1(R))}$. On the other hand, $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ is a 3-fold torsion theory for R-mod and so $\mathfrak{T}_2=\mathfrak{T}_{t_1(R)}$. Hence, by Lemma 1, we have that $t_1(R)+r(t_1(R))=R$.

Conversely, assume that $t_1(R)+r(t_1(R))=R$. Since $t_1(R)\cdot r(t_1(R))=0$, again by Lemma 1 we have that $r(t_1(R))$ is an idempotent two-sided ideal in R and $\mathfrak{T}_2=\mathfrak{T}_{t_1(R)}=\mathfrak{C}_{r(t_1(R))}$. From this it follows that $\mathfrak{T}_3=\mathfrak{T}_{r(t_1(R))}$ and hence $\mathfrak{T}_3$ is in fact a TTF-class.

Suppose now that $\mathfrak{T}_3$ is a TTF-class and that $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ has length 2. Then, by definition, $\mathfrak{C}_{t_1(R)}=\mathfrak{F}_{t_1(R)}$ and this also coincides with $\mathfrak{T}_{r(t_1(R))}$ by Lemma 1. Hence $r(t_1(R))\cdot t_1(R)=0$. Conversely, suppose that $\mathfrak{T}_3$ is a TTF-class and that $r(t_1(R))\cdot t_1(R)=0$. Then, by Lemma 1, $\mathfrak{T}_{r(t_1(R))}=\mathfrak{C}_{t_1(R)}$ and this also coincides with $\mathfrak{F}_{t_1(R)}$ again by Lemma 1. This shows that $\mathfrak{T}_1=\mathfrak{T}_3$ and thus $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ has length 2 by definition.

The last part of this proposition has already pointed out in [6, Corollary 1].

Proposition 3. *Let* $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ *be a 3-fold torsion theory for R-mod. Then* $\mathfrak{T}_1$ *is a* TTF-*class if and only if*

$$l(t_1(R))+t_1(R)=R.$$

Moreover, if this is the case, $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ *has length* 2 *if and only if* $t_1(R)\cdot l(t_1(R))=0$.

Proof. Suppose that $\mathfrak{T}_1$ is a TTF-class. Then there exists a class

$\mathfrak{T}$ of R-modules such that $(\mathfrak{T}, \mathfrak{T}_1, \mathfrak{T}_2)$ is also a 3-fold torsion theory for R-mod. If we denote by $t(M)$ the $\mathfrak{T}$-torsion submodule of an R-module M, then, by Proposition 2, $t(R)+r(t(R))=R$. Since $r(t(R))=t_1(R)$ and since $t(R)\cdot r(t(R))=0$, we have that $t(R)=l(t_1(R))$. Thus, $l(t_1(R))+t_1(R)=R$.

Conversely, assume that $l(t_1(R))+t_1(R)=R$. Since $l(t_1(R))\cdot t_1(R)=0$, it follows from Lemma 1 that $l(t_1(R))$ is an idempotent two-sided ideal in R and $\mathfrak{T}_{l(t_1(R))}=\mathfrak{C}_{t_1(R)}=\mathfrak{T}_1$. Thus, $\mathfrak{T}_1$ is in fact a TTF-class.

Suppose now that $\mathfrak{T}_1$ is a TTF-class and that $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ has length 2. Then, by definition, $\mathfrak{C}_{t(R)}=\mathfrak{T}_{t_1(R)}$ and hence $0=t_1(R)\cdot t(R)=t_1(R)\cdot l(t_1(R))$. Conversely suppose that $\mathfrak{T}_1$ is a TTF-class and that $t_1(R)\cdot l(t_1(R))=0$. Then, by Lemma 1, $\mathfrak{T}_{l(t_1(R))}=\mathfrak{C}_{t_1(R)}$ and this also coincides with $\mathfrak{F}_{t_1(R)}$ again by Lemma 1. This shows that $\mathfrak{T}_1=\mathfrak{T}_3$ and thus $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ has length 2 by definition.

Proposition 4. *Let $(\mathfrak{T}_1, \mathfrak{T}_2)$ be a hereditary torsion theory for R-*mod *such that any simple R-module belonging to $\mathfrak{T}_1$ has the projective cover. Then $\mathfrak{T}_2$ is a* TTF-*class if and only if there exists a projective R-module P with trace ideal I such that $\mathfrak{T}_2=\mathfrak{T}_I$.*

Proof. Let $\{S_\alpha\}_{\alpha\in A}$ be a complete set of representatives for the isomorphism classes of simple R-modules belonging to $\mathfrak{T}_1, P$ denotes the direct sum of projective covers of $S_\alpha, \alpha\in A$, and I denotes its trace ideal. Suppose that $\mathfrak{T}_2$ is a TTF-class. Then, by [5, Proposition 1], $\mathfrak{T}_1$ is closed under minimal epimorphisms and P belongs to $\mathfrak{T}_1$. Hence $\mathfrak{T}_2\subset\mathfrak{T}_I$.

If we assume that there is an R-module M such that $IM=0$, i.e., $\mathrm{Hom}_R(P, M)=0$, and that $t_1(M)\neq 0$. Then we can find an x $(\neq 0)$ in $t_1(M)$ and a simple R-module S belonging to $\mathfrak{T}_1$ such that

$$Rx \xrightarrow{f} S \longrightarrow 0$$

is exact. Let us denote by $P(S)$ the projective cover of S and by π the minimal epimorphism of $P(S)$ to S. Then there exists a homomorphism h of $P(S)$ to Rx such that $f\circ h=\pi$. We can extend h to a homomorphism h^* of P to M naturally, but by assumption $h^*=0$ and so $\pi=0$, a contradiction. This shows that $\mathfrak{T}_2=\mathfrak{T}_I$. Since the "if" part is clear, this completes the proof of the proposition.

Remark. It follows from this proposition that any hereditary 3-fold torsion theory $(\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3)$ for R-mod over a semiperfect ring R is determined by the trace ideal I of a certain projective R-module P. However, in this case we can show that

$$l(I)+I=R$$

and hence, by Proposition 3, $\mathfrak{T}_1$ is in fact a TTF-class. This result has already obtained by [5, Proposition 2].

To see this, let $e_1, e_2, \cdots, e_n$ be an orthogonal set of primitive idempotents of R whose sum is 1, the identity of R. We may assume that $I\neq 0$. Then $e_i\in I$ if and only if $e_iR=e_iI$, or equivalently, $e_iI\neq 0$. For, suppose that $e_iI\neq 0$. Then there exists an a ($\neq 0$) in e_iI. Since $Ra \subset I$, Ra belongs to $\mathfrak{T}_1=\mathfrak{C}_I$ and hence $Ia=I(Ra)=Ra$. So a is in Ia and we can find some x in I such that $a=xa$. Since $(1-e_ix)a=0$, if we assume that $e_iI\subset e_iN$, where N denotes the Jacobson radical of R, then e_ix is in N and hence $a=0$, a contradiction. Since e_iN is a unique maximal submodule of e_iR, e_iI must be equal to e_iR.

Now $I=e_1I+\cdots+e_nI$ and there exists some i such that $e_iI\neq 0$. So we may assume that $e_iI\neq 0$, $1\leqq i\leqq m$, and $e_iI=0$, $m+1\leqq i\leqq n$. Then $I=e_1I+\cdots+e_mI=e_1R+\cdots+e_mR=eR$, where $e=e_1+\cdots+e_m$, and so $l(I)=R(1-e)$. Thus we have $l(I)+I=R$.

Theorem 5. *Let P be a projective R-module with trace ideal I such that any simple R-module belonging to $\mathfrak{T}_I$ has the projective cover. Then*

(1) *$\mathfrak{F}_I$ is a* TTF-*class if and only if there exists a projective R-module Q with trace ideal $r(I)$ such that ${}_{R/I}Q$ is a generator for R/I*-mod.

(2) *If this is the case, then $(\mathfrak{C}_I, \mathfrak{T}_I, \mathfrak{F}_I)$ has length* 2 *if and only if $r(I)\cdot P=0$, and this is so if and only if ${}_{R/r(I)}P$ is a generator for $R/r(I)$*-mod.

Proof. Suppose that $\mathfrak{F}_I$ is a TTF-class. Then, by Proposition 4, there exists a projective R-module Q with trace ideal K such that $\mathfrak{F}_I =\mathfrak{T}_K$. Hence $\mathfrak{T}_I=\mathfrak{C}_K$ and $I+K=R$ by Lemma 1. Since K belongs to $\mathfrak{C}_K$, $IK=0$ and again by Lemma 1 we have $K=r(I)$. The rest of (1) follows from the same lemma.

(2) follows from Proposition 2 and Lemma 1. This completes the proof of the theorem.

Finally, we shall close the paper with the following theorem whose first half is due to [1, Proposition 11].

Theorem 6. *Let P be a projective R-module with trace ideal I. Then,*

(1) *$\mathfrak{C}_I$ is a* TTF-*class if and only if ${}_{R/l(I)}P$ is a generator for $R/l(I)$*-mod.

(2) *If this is the case and if we assume moreover that R is semiperfect, then there exists a projective R-module Q with trace ideal $l(I)$, and $(\mathfrak{C}_I, \mathfrak{T}_I, \mathfrak{F}_I)$ has length* 2 *if and only if $IQ=0$, and this is so if and only if ${}_{R/I}Q$ is a generator for R/I*-mod.

Proof. (1) By Proposition 3, $\mathfrak{C}_I$ is a TTF-class if and only if $l(I)+I=R$, and this is so if and only if $\mathfrak{T}_{l(I)}=\mathfrak{C}_I$ by Lemma 1. This means that ${}_{R/l(I)}P$ is a generator for $R/l(I)$-mod again by Lemma 1.

(2) Suppose that $\mathfrak{C}_I$ is a TTF-class and that R is semiperfect. Then, as was pointed out in the proof of [5, Proposition 2], there exists

a projective R-module Q with trace ideal K such that $\mathfrak{C}_I=\mathfrak{T}_K$. Hence we have $K=l(I)$. The rest of (2) follows from Lemma 1 and Proposition 3. This completes the proof of the theorem.

Added in proof. After submitting this paper, we became aware that Theorems 5 and 6 can be proved without restricted conditions. Its proof will appear somewhere.

References

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