133. On the Fundamental Units of Real Quadratic Fields

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1. Let $Q(\sqrt{D})$, ($D>0$ square-free rational integer), be a real quadratic field and put $D=n^2+r$ ($-n<r\leqq n$). Then, if $4n\equiv 0 \pmod r$ holds, the fundamental unit $\varepsilon_D>1$ of $Q(\sqrt{D})$ is well known ([1]) and such a real quadratic field $Q(\sqrt{D})$ is called *R-D* type. On the other hand, for any given real quadratic field $Q(\sqrt{D})$, its fundamental unit can be calculated by the continued fraction expansion of $\sqrt{D}$.

In this note, we shall first describe the fundamental units of all real quadratic fields in a similar fashion to *R-D* type, and give next its relation between contiriued fraction expansion. Finally, we shall give a generalization of a result of Morikawa [3] concerned with these facts.

2. The following theorem is a generalization of a result of Degert [1]:

Theorem 1. *For any given positive square-free integer D, let v_0 be the least positive integer such that $v_0^2D=n_0^2+r_0$ holds with integers n_0, r_0 satisfying $-n_0<r_0\leqq n_0$ and $4n_0\equiv 0 \pmod{r_0}$. Then the fundamental unit $\varepsilon_D>1$ of $Q(\sqrt{D})$ is of the following form:*

$\varepsilon_D=n_0+v_0\sqrt{D}$, $N\varepsilon_D=-\operatorname{sgn} r_0$ for $|r_0|=1$, (except for $D=5$, $v_0=1$),

$\varepsilon_D=(n_0+v_0\sqrt{D})/2$, $N\varepsilon_D=-\operatorname{sgn} r_0$ for $|r_0|=4$,

$\varepsilon_D=[(2n_0^2+r_0)+2n_0v_0\sqrt{D}]/|r_0|$, $N\varepsilon_D=1$ for $|r_0|\neq 1, 4$.

Remark. In the special case of $v_0=1$, this result coincides with Degert's.

Proof. Let $\varepsilon_D=(t_0+u_0\sqrt{D})/2$ be the fundamental unit of $Q(\sqrt{D})$ and ε_1 be the right-hand side of a formula for ε_D in Theorem 1. Then, it is easily shown that $u_0^2D=t_0^2\mp 4$, $4t_0\equiv 0 \pmod 4$ and that ε_1 is a unit of $Q(\sqrt{D})$. Here, if we suppose $\varepsilon_D\neq\varepsilon_1$, then it yields a contradiction. For, in the case of $|r_0|>4$, we get

$$\varepsilon_1=[(2n_0^2+r_0)+2n_0v_0\sqrt{D}]/|r_0|\geqq\varepsilon_D^2=(t_0^2\pm 2+t_0u_0\sqrt{D})/2.$$

Hence, we have $n_0v_0>t_0u_0$. On the other hand, since v_0 is the least positive integer such that $v_0^2D=n_0^2+r_0$, $-n_0<r_0\leqq n_0$, $4n_0\equiv 0 \pmod{r_0}$, we get $v_0<u_0$ and $n_0<t_0$, hence we have $n_0v_0<t_0u_0$. This is a contradiction. In other cases, we can easily induce contradiction similarly.

3. For any given D, it is generally difficult to find v_0 in Theorem 1, but if we use the continued fraction expansion of $\sqrt{D}$, v_0 is easily

obtained. In particular, if the length k of the period in the continued fraction expansion of $\sqrt{D}$ is even ($k=2m$), then v_0 in Theorem 1 is determined by the $(m-1)$th convergent in the continued fraction expansion of $\sqrt{D}$ as follows:

Theorem 2. *Let D be a positive square-free integer such that $D\not\equiv 5 \pmod 8$ and suppose that D has a prime divisor p such that $p\equiv 3 \pmod 4$. Let k be the length of the period in the regular continued fraction expansion of $\sqrt{D}$, A_ν/B_ν be its νth convergent and let $(\sqrt{D}+P_\nu)/Q_\nu$ be its νth complete quotient. Then, k is even ($k=2m$) and v_0 in Theorem 1 is equal to B_{m-1}. Moreover, $|r_0|$ in Theorem 1 is equal to Q_m which is equal to neither 1 nor 4 and the fundamental unit ε_D of $\boldsymbol{Q}(\sqrt{D})$ is of the following form:*

$$\varepsilon_D=[(2A_{m-1}^2+(-1)^{m-1}Q_m)+2A_{m-1}B_{m-1}\sqrt{D}]/Q_m, \qquad N\varepsilon_D=1.$$

Proof. From the assumption on D, it is easily proved that the length k of the period is even ($k=2$m) and that the fundamental unit ε_D of $\boldsymbol{Q}(\sqrt{D})$ is of the form $\varepsilon_D=t_0+u_0\sqrt{D}$, ($t_0$, u_0 integers). Hence, we have $\varepsilon_D=A_{k-1}+B_{k-1}\sqrt{D}$ and $N\varepsilon_D=1$. On the other hand, we have $Q_m\neq 1$ and the following relations (cf. [5]):

$$2A_{m-1}\equiv 2D\equiv 0 \pmod{Q_m},$$
$$B_{m-1}^2D=A_{m-1}^2+(-1)^{m-1}Q_m.$$

From these relations, we have $Q_m\neq 4$. Let ε_1 be the right-hand side of the formula for ε_D in Theorem 2, then ε_1 is a unit of $\boldsymbol{Q}(\sqrt{D})$ and ε_1 is equal to ε_D, since $1<\varepsilon_1<\varepsilon_D^2$. Therefore, v_0 in Theorem 1 is equal to B_{m-1} and $|r_0|=Q_m$.

4. As a sufficient condition for $Q_m=2$, we obtain

Theorem 3.[1] *Let $D=p$ or $2p$, where p is a prime number with $p\equiv 3 \pmod 8$ (resp. $\equiv 7 \pmod 8$). Let $k=2m$ be the even length of the period in the regular continued fraction expansion of $\sqrt{D}$ and A_ν/B_ν be its νth convergent. Then, $Q_m(=|r_0|)$ in Theorem 2 is equal to 2 and the fundamental unit ε_D of $\boldsymbol{Q}(\sqrt{D})$ is of the following form:*

$$\varepsilon_D=A_{m-1}^2+1+A_{m-1}B_{m-1}\sqrt{D} \text{ (resp. } A_{m-1}^2-1+A_{m-1}B_{m-1}\sqrt{D}), \qquad N\varepsilon_D=1.$$

Proof. Since $2D\equiv 0 \pmod{Q_m}$ and $D=p$ or $2p$, we have $Q_m=1, 2, 4, p, 2p$ or $4p$. On the other hand, $1<Q_m<\sqrt{D}$ and $Q_m\neq 4$ hold. Hence, we get $Q_m=2$. Thus, from Theorem 2, we have $\varepsilon_D=A_{m-1}^2\pm 1+A_{m-1}B_{m-1}\sqrt{D}$. Here, in the case of $p\equiv 3 \pmod 8$, $A_{m-1}^2-1+A_{m-1}B_{m-1}\sqrt{D}$ is not a unit, since $A_{m-1}^2-DB_{m-1}^2\not\equiv -2 \pmod 8$. Therefore, ε_D is equal to $A_{m-1}^2+1+A_{m-1}B_{m-1}\sqrt{D}$. Similarly, we can prove the other case.

Remark. In the case of $D=pq$, $(p<q)$, or $2pq$, $(2p<q)$, with $D\not\equiv 5 \pmod 8$, where p and q are odd prime numbers with p or $q\equiv 3 \pmod 4$, Nakahara shows in [4] that Q_m in Theorem 2 is equal to one

1) M. Yamauchi conjectured this fact and orally informed it to author.

of the three numbers 2, p and $2p$. Using this fact, he proves that the fundamental unit of $Q(\sqrt{D})$ has one of the following six forms: $A_{m-1}^2 \pm 1 + A_{m-1}B_{m-1}\sqrt{D}$, $\frac{2}{p}A_{m-1}^2 \pm 1 + \frac{2}{p}A_{m-1}B_{m-1}\sqrt{D}$, $\frac{1}{p}A_{m-1}^2 \pm 1 + \frac{1}{p}A_{m-1}B_{m-1}\sqrt{D}$.

In the case of real quadratic fields $Q(\sqrt{D})$ with $N\varepsilon_D = -1$, we can obtain similar result to Theorem 3 as follows:

Theorem 4. *Let $D = p_1$ or $2p_2$, where p_1 and p_2 are prime numbers with $p_1 \equiv 1 \pmod 8$ and $p_2 \equiv 5 \pmod 8$. Let $k = 2m+1$ be the odd length of the period in the regular continued fraction expansion of $\sqrt{D}$ and A_ν/B_ν be its νth convergent. Then, the fundamental unit ε_D is of the following form:*

$$\varepsilon_D = A_m B_m + A_{m-1}B_{m-1} + (B_m^2 + B_{m-1}^2)\sqrt{D}, \qquad N\varepsilon_D = -1.$$

Proof. Let $\sqrt{D} = [b_0, \overline{b_1, \cdots, b_k}]$ be the regular continued fraction expansion of $\sqrt{D}$, where k is the length of the period. From the condition on D, it is evident that k is odd ($k = 2m+1$) and $\varepsilon_D = A_{k-1} + B_{k-1}\sqrt{D}$. On the other hand, it is well known that $b_1, \cdots, b_{k-1}$ are symmetric: $b_{k-\nu} = b_\nu$, $(1 \leqq \nu \leqq k-1)$. Hence, we get $A_{k-1} = A_m B_m + A_{m-1}B_{m-1}$ and $B_{k-1} = B_m^2 + B_{m-1}^2$. Therefore, we have the Theorem 4.

5. Finally we give a generalization of Morikawa's result from our view-point.

Theorem 5.[2)] *For any positive integer $a > 0$, put $a^2 \pm 2 = b^2 D$, where D is square-free. If $D \neq 2, 3$, and 6, and if at least one of the following conditions (α) and (β) is satisfied, then $Q_m (= |r_0|)$ in Theorem 2 is equal to 2 and $\varepsilon = a^2 \pm 1 + ab\sqrt{D}$ is the fundamental unit of $Q(\sqrt{D})$:*

(α) *$a < (2D-1)\sqrt{D-2}$ or $b < 2D-3$,*

(β) *$a = p^k$ or $2p^k$, where p is a prime number and k is a positive integer.*

Proof. Let $\varepsilon = (t + u\sqrt{D})/2 > 1$ be a unit of $Q(\sqrt{D})$ with $N\varepsilon = 1$. Put $\varepsilon^n = (t_n + u_n\sqrt{D})/2$, $(n \geqq 1)$. Then t_n is a monic polynomial of t with integral coefficients and has the following properties:

(i) t_n is a monotonically increasing function of t,

(ii) $t_n - 2 = (t-2)\{(t-2)^{(n-1)/2} + \cdots + \frac{1}{24}(n^3-n)(t-2) + n\}^2$ for odd n,

(iii) $t_n + 2 = (t+2)\{(t+2)^{(n-1)/2} - \cdots \pm \frac{1}{24}(n^3-n)(t+2) \mp n\}^2$ for odd n.

From these facts, we can prove our Theorem 5 immediately.

2) Morikawa [2] proved this theorem in the special case that a is a prime number.

References

[1] G. Degert: Über die Bestimmung der Grundeinheit gewisser reell-quadratischer Zahlkörper. Abh. Math. Sem. Univ. Hamburg, **22**, 92–97 (1958).

[2] M. Kutsuna: On the fundamental units of a certain type of real quadratic fields (Informal notes in Japanese). Suron Hanti, **1**, 116–138 (1971).

[3] R. Morikawa: On the fundamental units of certain real quadratic number fields (to appear).

[4] T. Nakahara: On the fundamental units and an estimate of the class numbers of real quadratic fields. Rep. Fac. Sci. Eng. Saga Univ., **1**, 104–116 (1973).

[5] O. Perron: Die Lehre von den Kettenbrüchen, Band I. Teubner Verlag (1954).