# 130. On C. Loncour's Results

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The purpose of this note is to present a theorem which refines the results of C. Loncour [3, Theorems 5 and 6] with a simple proof. Let $G$ be the direct product of finite groups $M$ and $M'$, $KG$ the group algebra of $G$ over a field $K$ of characteristic $p$, $J(KG)$ the radical of $KG$, $t(G)$ the nilpotency index of $J(KG)$. Let $r_i=[J(KM)^i:K]$ (the $K$-dimension of $J(KM)^i$) and $r'_i=[J(KM')^i:K]$, where $J(KM)^0=KM$ and $J(KM')^0=KM'$. Let $s$ be a fixed integer such that $1\leqq s<t(G)$. Let $T_i=\{a_{ij}\mid 1\leqq j\leqq r_i-r_{i+1}\}$ be a subset of $J(KM)^i$ which forms a $K$-basis of $J(KM)^i$ modulo $J(KM)^{i+1}$ (for $i<s$), and $T_s=\{a_{sk}\mid 1\leqq k\leqq r_s\}$[1] a $K$-basis of $J(KM)^s$. Quite similarly, we define $T'_k=\{a'_{kl}\mid 1\leqq l\leqq r'_k-r'_{k+1}\}$ and $T'_s=\{a'_{sl}\mid 1\leqq l\leqq r'_s\}$ for $KM'$.

Now, our theorem is stated as follows:

**Theorem.**[2] (1) $[J(KG)^s:K]=\sum_{i=0}^{s} r_i r'_{s-i}-\sum_{i=1}^{s} r_i r'_{s-i+1}$.

(2) $J(KG)^s=\sum_{i=0}^{s} J(KM)^i J(KM')^{s-i}$.

(3) $t(G)=t(M)+t(M')-1$.

(4)[3] $\Omega=\bigcup_{s\leqq i+k} T_i T'_k$ *forms a basis for* $J(KG)^s$, *where* $T_iT'_k=\{tt'\mid t\in T_i, t'\in T'_k\}$.

**Proof.** (1) and (2): We assume first that $s=1$. Let $L$ be a splitting field for $G$ and a finite dimensional separable extension of $K$. Then $L$ is a splitting field for $M$ and $M'$, $[J(KG):K]=[J(LG):L]$ (cf. [2, p. 252]), and $\{U_i\otimes V_j\mid 1\leqq i\leqq a, 1\leqq j\leqq b\}$ is the set of all irreducible $LG$-modules (cf. [1, p. 586]), where $\{U_i\mid 1\leqq i\leqq a\}$ and $\{V_j\mid 1\leqq j\leqq b\}$ are the sets of all irreducible $LM$-modules and $LM'$-modules, respectively. Thus, $r_1r'_0+r_0r'_1-r_1r'_1=r'_0(r_0-\sum_{i=1}^{a}[U_i:L]^2)+r_0(r'_0-\sum_{i=1}^{b}[V_i:L]^2)-(r_0-\sum_{i=1}^{a}[U_i:L]^2)(r'_0-\sum_{i=1}^{b}[V_i:L]^2)=r_0r'_0-\sum_{i=1}^{a}\sum_{j=1}^{b}[U_i:L]^2[V_j:L]^2=[J(LG):L]=[J(KG):K]$, which proves (1) for the case $s=1$. Noting that $J(KM)M'\cap J(KM')M=J(KM)J(KM')\cong J(KM)\otimes J(KM')$, we can see that $[(J(KM)M'+J(KM')M):K]=[J(KM)M':K]+[J(KM')M:K]-[(J(KM)M'\cap J(KM')M):K]=r_1r'_0+r_0r'_1-r_1r'_1=[J(KG):K]$. Since $J(KM)M'+J(KM')M$ is a nilpotent ideal whose $K$-dimension is $[J(KG):K]$, we have $J(KG)=J(KM)M'+J(KM')M$, which proves

1) If $J(KM)^t=0$ for some $t<s$, then we set $T_j=\phi$ and $r_j=0$ for $j\geqq t$.

2) Cf. [2, pp. 122–123 and 251–254].

3) $\Omega$ is slightly different from C. Loncour's basis of [3, Theorem 5 (2)]. However, it is easy to give his basis for $J(KG)$ by Theorem (4).

(2) for the case $s=1$. Now, $J(KG)^s=(J(KM)M'+J(KM')M)^s =\sum_{i=0}^{s} J(KM)^i J(KM')^{s-i}$ implies (2). Noting that $(\sum_{k=i}^{s} J(KM)^k J(KM')^{s-k}) \cap J(KM)^{i-1}J(KM')^{s-i+1}=J(KM)^i J(KM')^{s-i+1}$, we can prove (1) by (2).

(3): The assertion of (3) is easily seen by (2).

(4): Since the cardinal number of $\Omega$ is $[J(KG)^s; K]$, it remains only to prove that $\Omega$ is independent over $K$. Assume that $\sum_{i,j,k,l} \alpha_{ijkl} a_{ij} a'_{kl}=0$, where $\alpha_{ijkl}$ are elements of $K$. We set $\eta_m =\sum_{i,j,l} \alpha_{ijml} a_{ij} a'_{ml}$ and $\omega_n=\sum_{q=n}^{s} \eta_q$. Then $\omega_n+\sum_{q<n} \eta_q=0$, $\eta_m$ is an element of $J(KM)^{s-m}J(KM')^m$ and $\omega_n$ is an element of $KMJ(KM')^n$. Since $\eta_0=-\omega_1$ is an element of $J(KM)^s M' \cap J(KM')M=J(KM)^s J(KM')$ and $(J(KM)^s \otimes KM')/(J(KM)^s \otimes J(KM'))$ is naturally isomorphic to $J(KM)^s \otimes KM'/J(KM')(x\otimes y \bmod J(KM)^s \otimes J(KM') \rightarrow x\otimes(y \bmod J(KM')))$, it follows that all coefficients of $\eta_0$ are zero. Thus, $\eta_1=-\omega_2$ is an element of $(J(KM)^{s-1}J(KM')) \cap (KM\cdot J(KM')^2)=J(KM)^{s-1}J(KM')^2$ and by considering the natural isomorphism $(J(KM)^{s-1}\otimes J(KM'))/(J(KM)^{s-1} \otimes J(KM')^2) \cong J(KM)^{s-1}\otimes J(KM')/J(KM')^2$, we see that all coefficients of $\eta_1$ are zero. Repeating the above procedure, we obtain eventually $\alpha_{ijkl}=0$ for all $i, j, k, l$.

**Corollary.** *Let $R$ be a semi-primary ring such that the center of $R/J(R)$ contains the prime field of characteristic $p$. Then $J(RG) =J(RM)M'+J(RM')M+J(R)G$.*

**Proof.** The assertion is clear by Theorem (2) and making the same method as in [4, Lemma 1].

## References

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