129. *Fundamental Solution of Partial Differential Operators of Schrödinger's Type.* I

By Daisuke FUJIWARA
Department of Mathematics, University of Tokyo

(Comm. by Kôsaku YOSIDA, M. J. A., Oct. 12, 1974)

§ 1. **Preliminaries.** Let $ds^2=\sum_{ij}^{n} g_{ij}(x)dx_i dx_j$ be a Riemannian metric on R^n. The Laplacian $\Delta=\frac{1}{\sqrt{g}}\sum_{ij}\frac{\partial}{\partial x_i}\left(\sqrt{g}\,g^{ij}\frac{\partial}{\partial x_j}\right)$ associated with this metric naturally defines a self-adjoint operator in $L^2(R^n)$ with respect to the measure $\sqrt{g}\,dx$. This generates a one parameter group of unitary operators $U_t=\exp\frac{1}{2}i\nu^{-1}\Delta t, \nu>0, t\in R$. For any f in the domain of Δ, the function $u=U_t f$ satisfies the following equations

(1) $$\left(i\nu\frac{\partial}{\partial t}+\frac{1}{2}\Delta\right)u=0 \qquad \text{for any } t\in R,$$

(2) $$s-\lim_{t\to 0} U_t f=f.$$

The aim of this note is to construct, under assumptions in §§ 2 and 3, the distribution kernel $U(t,x,y)$ of the operator U_t. Our proof follows Feynman's idea [2]. Combining technique of Calderòn-Vaillancourt with method of oscillatory integrals [4], we can give rigorous mathematical reasoning to Feynman's idea.

§ 2. **Parametrix.** Let us denote by $q=q(t,y,\eta)$ and $p=p(t,y,\eta)$ the solution of the Hamiltonian equations

(3) $$\frac{dq}{dt}=\frac{\partial H}{\partial p}, \qquad \frac{dp}{dt}=-\frac{\partial H}{\partial q}$$

satisfying initial conditions at $t=0$; $q=y$ and $p=\eta$, where H is the Hamiltonian function $H(q,p)=\frac{1}{2}\sum_{ij} g^{ij}(q)p_i p_j$. Since H is a homogeneous function of p's, we have

(4) $$q(t,y,\eta)=q(1,y,t\eta) \quad \text{and} \quad tp(t,y,\eta)=p(1,y,t\eta).$$

Our first assumption is that

(A.I) the canonical transformation $\chi_t:(x^\circ,\eta)\mapsto(x,\xi)=(q(t,x^\circ,\eta), p(t,x^\circ,\eta))$ induces global diffeomorphism of the base space R^n.

The generating function of this canonical transformation is

(5) $$S_0(t,x,\eta)=\int_0^t L(q,\dot{q})ds+x^\circ\cdot\eta,$$

where $L(q,\dot{q})$ is Lagrangean corresponding to Hamiltonian H and the integral should be taken along the classical orbit from x°. $x^\circ = x^\circ(t, x, \eta)$ is the unique solution of the equation $x = q(t, x^\circ, \eta)$. We set

(6) $$S(t, x, \xi, y) = S_0(t, x, \xi) - \xi \cdot y.$$

Our parametrix is of the form

(7) $$E_N(t, x, \xi, y) = \exp i\nu S(t, x, \xi, y) e(t, x, \xi)$$

with

(8) $$e(t, x, \xi) = \sum_{r=0}^{N} (i\nu)^{-r} e_r(t, x, \xi),$$

where N will be fixed later. Amplitude functions $e_r(t, x, \xi)$ are determined inductively by

(9) $$\frac{D}{Dt} e_{r+1} + \frac{1}{2} \Delta S e_{r+1} + \frac{1}{2} \Delta e_r = 0, \qquad e_{-1} = 0,$$

with initial conditions $e_0(0, x, \xi) = 1$ and $e_r(0, x, \xi) = 0$. Here $\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{n} \dot{q}_j \frac{\partial}{\partial q_j}$. Thus we have

(10) $$e_0(t, x, \xi) = (g(x)/g(x^\circ(t, x, \xi)))^{1/4} \quad \text{and}$$
$$e_r(t, x, \xi) = -e_0(t, x, \xi) \int_0^t \frac{1}{2} \Delta_z e_{r-1}(s, z(s), \xi)/e_0(s, z(s), \xi) ds,$$

where $z(s) = q(s, x^\circ(t, x, \xi), \xi)$. Our parametrix satisfies

(11) $$\left(\nu i \frac{\partial}{\partial t} + \frac{1}{2} \Delta\right) E_N(t, x, \xi, y) = (\nu i)^{-N} \frac{1}{2} \Delta e_N(t, x, \xi) \exp i\nu S.$$

Later we use homogeneity property;

(12) $$\left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta e_j(t, x, \xi) = t^{j+|\beta|} \left(\frac{\partial}{\partial x}\right)^\alpha \left(\frac{\partial}{\partial \xi}\right)^\beta e_j(1, x, t\xi).$$

§ 3. Assumptions.

We assume the following assumptions (A–II) ~(A–VI) as well as (A–I) in the previous section. (A–II) there exists a constant $C_0 > 0$ such that we have $C_0 \leqq \left(\sum_{ij} g_{ij}(x)\xi_i\xi_j\right) \Big/ \left(\sum_{ij} g_{ij}(y)\xi_i\xi_j\right) \leqq C_0^{-1}$ for any x, y in R^n, $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in R^n$. (A–III) for any multi-index α, there exists a constant $C_\alpha > 0$ such that we have $\left|\left(\frac{\partial}{\partial x}\right)^\alpha g_{ij}(x)\right| \leqq C_\alpha$, for any $x \in R^n$. (A–IV) there exists a constant $C_2 > 0$ such that we have $|\mathrm{grad}_\xi (S_0(t, x, \xi) - S_0(t, z, \xi))| \geqq C_2 |x - z|$ and $|\mathrm{grad}_y (S_0(t, y, \xi) - S_0(t, y, \eta))| \geqq C_2 |\xi - \eta|$ for any $t \in [0, T]$, $x, z, y \in R^n$ and ξ, η in R^n. (A–V) for any multi-index $\alpha, |\alpha| \geqq 2$, there exists a constant $C > 0$ such that we have $\left|\left(\frac{\partial}{\partial \xi}\right)^\alpha (S_0(t, x, \xi) - S_0(t, z, \xi))\right| \leqq C|x - z|$ and $\left|\left(\frac{\partial}{\partial y}\right)^\alpha (S_0(t, y, \xi) - S_0(t, y, \eta))\right| \leqq C |\xi - \eta|$ for any t in $[0, T]$ and $x, z, y \in R^n$ and $\xi, \eta \in R^n$. (A–VI) for any multi-indices α, β, there exists a constant $C > 0$ such that we have $\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial \xi}\right)^{\beta} e_0(t, x, \xi)\right| \leqq C$ for any $t \in [0, T]$, $x \in R^n$, $\xi \in R^n$.

Remark. 1) These assumptions may be redundant. 2) Assumption (A–I) is very restrictive. We can use Maslov's theory of canonical operators and replace (A–I) with less restrictive assumption.

§ 4. Results. We define two integral transformations;

$$E_N(t)f(x) = (\nu/2\pi)^n \iint_{R^{2n}} E_N(t, x, \xi, y) f(y) dy d\xi, \tag{13}$$

$$\begin{aligned} &F_N(t)f(x) \\ &= (i\nu)^{-N}(\nu/2\pi)^n \iint_{R^{2n}} \frac{1}{2} \Delta e_N(t, x, \xi) \exp i\nu S(t, x, \xi, y) f(y) dy d\xi. \end{aligned} \tag{14}$$

These are well defined for functions $f(x)$ in $C_0^\infty(R^n)$. For the sake of brevity we shall omit writing domains of integration if there is no fear of confusion.

Theorem 1. *The equality* (13) *naturally defines a bounded linear operator* $E_N(t)$, $t \in [0, T]$, *in* $L^2(R^n)$ *with respect to the measure* $\sqrt{g}\, dx$.

Theorem 2. *We have*

$$\lim_{k \to \infty} \left\| E_N\left(\frac{T}{k}\right) E_N\left(\frac{T}{k}\right) \cdots E_N\left(\frac{T}{k}\right) - \exp i\nu^{-1} T \frac{1}{2} \Delta \right\| = 0. \tag{15}$$

cf. R. Feyman [2].

§ 5. Outline of proof. From (10) and (A–VI) we see all of $e_r(t, x, \xi)$ enjoy the same estimate as $e_0(t, x, \xi)$.

Lemma. *Assume that* $a(x, \xi)$ *is a function in* $C^\infty(R^{2n})$ *and that for any multi-indices* α, β *there exists a constant* C *such that we have*

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial \xi}\right)^{\beta} a(x, \xi)\right| \leqq C \qquad \textit{for any } x \in R^n, \xi \in R^n. \tag{16}$$

Define a mapping A as

$$Af(x) = \iint_{R^{2n}} a(x, \xi) \exp i\nu S(t, x, \xi, y) f(y) dy d\xi \tag{17}$$

for any f in $C_0^\infty(R^n)$. Then there exists a constant $C > 0$ such that we have

$$\|Af\| \leqq C\nu^{-n} \|f\|, \tag{18}$$

where $\| \quad \|$ is the L^2-norm and $C > 0$ is independent of t, ν and f (cf. [3]).

Theorem 1 follows from this lemma and (15). If we use this lemma for $a(x, \xi) = \Delta e_N(1, x, \xi)$ we have

$$\|F_N(t)f\| \leqq Ct^N \nu^{-N} \|f\|. \tag{19}$$

Equality (11) implies that

$$E_N(t) = \exp i\nu \frac{1}{2} t\Delta + R_N(t), \ R_N(t) = \int_0^t \exp \nu i \frac{1}{2}(t-s) \Delta F_N(s) ds. \tag{20}$$

(19) and (20) mean that

$$\|R_N(t)\| \leqq Ct^{N+1} \nu^{-N}. \tag{21}$$

We have for k products of operators $E_N\left(\frac{T}{k}\right) E\left(\frac{T}{k}\right) \cdots E_N\left(\frac{T}{k}\right)$

$=\left(\exp \nu i\frac{1}{2}\frac{T}{k}\Delta+R_N\left(\frac{T}{k}\right)\right)\cdots\left(\exp i\nu\frac{1}{2}\frac{T}{k}\Delta+R_N\left(\frac{T}{k}\right)\right)$. Since $\exp i\frac{1}{2}\nu t\Delta$ is unitary, $\left\|E_N\left(\frac{T}{k}\right)E_N\left(\frac{T}{k}\right)\cdots E_N\left(\frac{T}{k}\right)-\exp i\frac{1}{2}\nu T\Delta\right\| \leqq \sum_{l=1}^{k}\binom{k}{l}\left\|R_N\left(\frac{T}{k}\right)\right\|^l=\left(1+\left\|R_N\left(\frac{T}{k}\right)\right\|\right)^k-1$. This and (21) prove Theorem 2 if we choose $N\geqq 1$.

References

[1] A. P. Calderòn and R. Vaillancourt: On the boundedness of pseudo-differential operators. Jour. Math. Soc. Japan, **23**, 374–378 (1971).
[2] R. Feynman: Space-time approach to nonrelativistic quantum mechanics. Reviews of Modern Physics., **20**, 367–384 (1948).
[3] D. Fujiwara: On the boundedness of integral transformations with highly oscillatory kernel (to appear).
[4] P. D. Lax: Asymptotic solutions of oscillatory initial value problems. Duke Math. Jour., **24**, 627–646 (1957).
[5] V. P. Maslov: Théorie des perturbations et méthodes asymptotiques (French translation). Paris (1972).