124. *Hypoelliptic Differential Operators with Double Characteristics*

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(Comm. by Kôsaku YOSIDA, M. J. A., Oct. 12, 1974)

In this note, we shall consider the hypoellipticity of the following operator in $\boldsymbol{R}^2$:

$$P(x,t,D_x,\partial_t)=(\partial_t+taD_x)(\partial_t+tbD_x)+cD_x \\ +A(x,t)tD_x+B(x,t),$$

where $\partial_t=\partial/\partial t$, $D_x=-i\partial/\partial x$ and $a,b,c\in\boldsymbol{C}$ and $A(x,t)$, $B(x,t)\in C^\infty(\boldsymbol{R}^2)$. (Cf. Grušin [1], [2], Sjöstrand [3], Treves [4].) A linear (pseudo-) differential operator $Q(x,D_x)$ in $\boldsymbol{R}^n$ is called hypoelliptic in an open subset $\Omega\subset\boldsymbol{R}^n$ if

$$\text{sing supp}\, u=\text{sing supp}\, Qu, \qquad u\in\mathcal{E}'(\Omega).$$

If $A\equiv 0$ and $B\equiv 0$, then we have

Theorem 0 (cf. [1], Theorem 1.2). *Assume that* $\mathrm{Re}\, a\cdot\mathrm{Re}\, b<0$. *Then*

$$P_1(x,t,D_x,\partial_t)=(\partial_t+taD_x)(\partial_t+tbD_x)+cD_x$$

is hypoelliptic in $\boldsymbol{R}^2$ *if and only if*

$$\frac{c}{b-a}\notin\boldsymbol{Z}.$$

Thus, in this note, we assume that

$$\text{(A)}\qquad \mathrm{Re}\, a<0,\ \mathrm{Re}\, b>0,\ \frac{c}{b-a}\in\boldsymbol{Z}^+\cup\{0\}.$$

We shall give the *sufficient* conditions on A, B for P to be hypoelliptic in a neighbourhood of $(x,t)=(0,0)$ (see Corollary 1 and Corollary 2 below). The case that $\mathrm{Re}\, a>0$, $\mathrm{Re}\, b<0$, $c/(b-a)\in\boldsymbol{Z}^+\cup\{0\}$ can be proved in exactly the same way. Now we state the main result:

Theorem 1 (cf. [3], Proposition 5.4). *Under the assumption* (A), *there exist properly supported operators*

$$\mathcal{P}=\begin{pmatrix}P, & R^-\\ R^+, & 0\end{pmatrix}: \begin{matrix}\mathcal{D}'(\boldsymbol{R}^2)\\ \oplus\\ \mathcal{D}'(\boldsymbol{R})\end{matrix}\to\begin{matrix}\mathcal{D}'(\boldsymbol{R}^2)\\ \oplus\\ \mathcal{D}'(\boldsymbol{R})\end{matrix}$$

$$\mathcal{G}=\begin{pmatrix}G, & G^+\\ G^-, & G^{-+}\end{pmatrix}: \begin{matrix}\mathcal{D}'(\boldsymbol{R}^2)\\ \oplus\\ \mathcal{D}'(\boldsymbol{R})\end{matrix}\to\begin{matrix}\mathcal{D}'(\boldsymbol{R}^2)\\ \oplus\\ \mathcal{D}'(\boldsymbol{R})\end{matrix}$$

with the following properties:

(i) $\mathcal{G}\cdot\mathcal{P}-I$ *and* $\mathcal{P}\cdot\mathcal{G}-I$ *have* C^∞ *kernels.*

(ii) *For all* $s\in\boldsymbol{R}$

$$G: H_s^{\mathrm{loc}}(\boldsymbol{R}^2)\to H_{s+1}^{\mathrm{loc}}(\boldsymbol{R}^2),$$

$$G^+ : H_s^{\mathrm{loc}}(\boldsymbol{R}) \to H_{s+1}^{\mathrm{loc}}(\boldsymbol{R}^2),$$
$$G^- : H_s^{\mathrm{loc}}(\boldsymbol{R}^2) \to H_s^{\mathrm{loc}}(\boldsymbol{R}),$$
$$G^{-+} : H_s^{\mathrm{loc}}(\boldsymbol{R}) \to H_s^{\mathrm{loc}}(\boldsymbol{R})$$

are continuous.

(iii) $WF'(G) \subset \{((x,t,\xi,\tau),(x,t,\xi,\tau)) \in (T^*(\boldsymbol{R}^2)\backslash 0) \times (T^*(\boldsymbol{R}^2)\backslash 0)\}$, $WF'(R^-), WF'(G^+) \subset \{((x,0,\xi,0),(x,\xi)) \in (T^*(\boldsymbol{R}^2)\backslash 0) \times (T^*(\boldsymbol{R})\backslash 0)\}$, $WF'(R^+), WF'(G^-) \subset \{((x,\xi),(x,0,\xi,0)) \in (T^*(\boldsymbol{R})\backslash 0) \times (T^*(\boldsymbol{R}^2)\backslash 0)\}$, $WF'(G^{-+}) \subset \{((x,\xi),(x,\xi)) \in (T^*(\boldsymbol{R})\backslash 0) \times (T^*(\boldsymbol{R})\backslash 0)\}$.

Remark 1. It follows from the assumption (A) that the principal symbol of G^{-+} is equal to 0 for $\xi > 0$ and $(2m+1)(b-a)C_m$ for $\xi < 0$ where $c = m(b-a)$ with $m \in \boldsymbol{Z}^+ \cup \{0\}$ and C_m is a non zero constant. Thus G^{-+} is elliptic for $\xi < 0$.

Remark 2. It follows from (i) that $G^{-+} \equiv -G^-PG^+ \bmod C^\infty$ kernel. Hence we see from (iii) that the problem of the location of the singularities for P in the *characteristic* $\{(x,0,\xi,0) \in (T^*(\boldsymbol{R}^2)\backslash 0)\}$ can be reduced to the same problem for G^{-+}. In fact we can prove

Theorem 2 (cf. [2], Theorem 4.2). *P is hypoelliptic in a neighbourhood of $(x,t) = (0,0)$ if and only if G^{-+} is hypoelliptic in a neighbourhood of $x = 0$.*

In the case that $a + \bar{b} = 0$, $c = 0$, calculating the symbols of G^{-+} explicitly*) and using Theorem 2, we obtain the following corollaries.

Corollary 1. *Let $a + \bar{b} = 0$, let $c = 0$, let $A(x,t) = \omega(x)t^j$ where $\omega(x) \in C^\infty(\boldsymbol{R})$ and $j \in \boldsymbol{Z}^+ \cup \{0\}$, and let $B(x,t) \equiv 0$. If $\omega(x) \neq 0$ in a neighbourhood of $x = 0$, then P is hypoelliptic in a neighbourhood of $(x,t) = (0,0)$.*

Remark 3. Similarly we can prove the following result (see [4], Example II. 5.2): Under the assumption that $h(0) = 0$,

$$P = \left(\partial_t - \frac{1}{2}tD_x\right)\left(\partial_t + \frac{1}{2}tD_x\right) + h(t)D_x$$

is hypoelliptic in $\boldsymbol{R}^2$ if $h(t)$ does not vanish of infinite order at $t = 0$. In fact, putting $a = -1/2$, $b = 1/2$, $A(x,t) = h(t)/t$ and $B(x,t) \equiv 0$, we find that if $h(t)$ does not vanish of infinite order at $t = 0$, then G^{-+} is elliptic for $\xi > 0$, which proves that G^{-+} is hypoelliptic in $\boldsymbol{R}$ (see Remark 1 and [2], Theorem 4.3).

Corollary 2. *Let $a = -1$, let $b = 1$, let $c = 0$, let $A(x,t) = \omega(x)t$ where $\omega(x) \in C^\infty(\boldsymbol{R})$ and let $B(x,t) = \varepsilon(x)t^2$ where $\varepsilon(x) \in C^\infty(\boldsymbol{R})$. Assume that $\omega(x)$ has a zero of finite order k at $x = 0$. If $k \geq 2$ and there exists a constant $C > 0$ such that in a neighbourhood of $x = 0$*

$$C\left|\mathrm{Im}\left(\frac{\beta(x)}{\alpha(x)}\right)\right| > \left|\mathrm{Re}\left(\frac{\beta(x)}{\alpha(x)}\right)\right|$$

*) (Added in proof.) Cf. Boutet de Monvel and Trèves [5], §8.

where $\alpha(x)=\omega(x)/2$ *and* $\beta(x)=\varepsilon(x)/2+\omega^2(x)/8-D_x\omega(x)/4$, *then* P *is hypoelliptic in a neighbourhood of* $(x,t)=(0,0)$.

The details will be given somewhere else.

References

[1] Grušin, V. V.: On a class of hypoelliptic operators. Math. USSR Sbornik, **12**, 458–476 (1970).

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