164. Defect Relations and Ramification

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In this paper we generalize the theory of ramified values in the Nevanlinna theory ([4], [7]) to the case of equidimensional holomorphic maps from C^n into projective algebraic manifolds and we prove variants of a defect relation of Carlson and Griffiths [1]. (See also [3], [9].)

1. Let W be a projective algebraic manifold of dimension n and L a line bundle on W. Iitaka [5] defined the *L-dimension* $\kappa(L, W)$ of W, which is roughly the polynomial order of $\dim H^0(W, \mathcal{O}(mL))$ as a function of m, as follows. If there is a positive integer m_0 such that $\dim H^0(W, \mathcal{O}(m_0L))>0$, we have the following estimate:

$$\alpha m^\kappa \leqq \dim H^0(W, \mathcal{O}(mm_0L)) \leqq \beta m^\kappa,$$

for large integer m and positive constants α, β, where κ is a non-negative integer uniquely determined by L. Then we define $\kappa(L, W)=\kappa$. In the other case, we put $\kappa(L, W)=-\infty$. In particular, $\kappa(L, W)=n$ if and only if

$$\limsup_{m\to+\infty} m^{-n} \dim H^0(W, \mathcal{O}(mL))>0.$$

For a divisor D on W, denote by $[D]$ the line bundle associated with D. Define $\kappa(D, W)=\kappa([D], W)$. By $L_1+\cdots+L_k$, we mean the tensor product $L_1\otimes\cdots\otimes L_k$ of line bundles $L_1, \cdots, L_k$. Moreover we shall consider linear combinations of line bundles: $L=q_1L_1+\cdots+q_kL_k$, with rational numbers $q_1, \cdots, q_k$. Define $\kappa(L, W)$ to be $\kappa(mL, W)$ for any positive integer m such that each mq_i is an integer.

2. We shall consider holomorphic maps $f: C^n \to W$, and assume that f is *non-degenerate*, i.e., the Jacobian J_f of f does not vanish identically. Let D be an effective divisor on W. Denote by $\operatorname{Supp}(f^*D)$ the support of the divisor f^*D. Namely, if $f^*D=\sum_s m_sZ_s$, with Z_s irreducible, we put $\operatorname{Supp}(f^*D)=\sum_s Z_s$. Let $(z_1, \cdots, z_n)$ be holomorphic coordinates in C^n, and let $B[r]$ denote a ball of radius r: $B[r]=\{z\in C^n \mid \|z\|<r\}$, where $\|z\|^2=|z_1|^2+\cdots+|z_n|^2$. For a set X in C^n, let $X[r]=X\cap B[r]$. We use the following notations:

$$\psi=(2\pi)^{-1}\sqrt{-1}\partial\bar\partial \log \|z\|^2,$$

$$N(D, r)=\int_0^r \left(\int_{f^*D[t]} \psi^{n-1}\right) t^{-1}dt,$$

$$\bar{N}(D, r)=\int_0^r \left(\int_{\operatorname{Supp}(f^*D)[t]} \psi^{n-1}\right) t^{-1}dt,$$

$$N_1(r)=\int_0^r\left(\int_{(Jf)[t]}\psi^{n-1}\right)t^{-1}dt.$$

Definition. Let L be a line bundle on W and let ω be a real $(1,1)$ form belonging to the Chern class $c_1(L)$. Set

$$T(L,r)=\int_0^r\left(\int_{B[t]}f^*\omega\wedge\psi^{n-1}\right)t^{-1}dt.$$

For a divisor D, set $T(D,r)=T([D],r)$. For a linear combination of line bundles: $L=q_1L_1+\cdots+q_kL_k$, set $T(L,r)=q_1T(L_1,r)+\cdots+q_kT(L_k,r)$.

Let D be an effective divisor on W. We have the following

Theorem 1 (see [1], [3] for a proof).

(1) $$N(D,r)<T(D,r)+O(1).$$

Proposition 1. *If $f(C^n)\cap D\neq\emptyset$, then*

$$\liminf_{r\to+\infty}[T(D,r)/\log r]>0\cdot$$

Corollary. *If L is a line bundle such that $\kappa(L,W)\geqq 1$, then*

(2) $$\liminf_{r\to+\infty}[T(L,r)/\log r]>0.$$

Proposition 2. *If L is a line bundle such that $\kappa(L,W)=n$, then*

(3) $$\liminf_{r\to+\infty}[T(L,r)/T(D,r)]>0.$$

Proof. This follows from the fact that there is an effective divisor $Z\in|mL-D|$ for some large integer m (cf. [6], [8]).

Let $D=D_1+\cdots+D_k$ be a divisor on W satisfying the following conditions:

(4) (i) Each D_i is non-singular,
(ii) D has only normal crossings.

By a similar method as in [1], [3], [8], we obtain

Theorem 2 (Second main theorem). *Let L be a line bundle such that $\kappa(L,W)=n$, β a constant, $0<\beta<1$, and let K_W denote the canonical bundle of W. Then*

(5) $$T(D,r)-N(D,r)+N_1(r)\leqq -T(K_W,r)+O(\log T(L,r)),$$

for $r\notin E$, where E is a union of intervals $\subset[0,+\infty)$ such that $\int_E d(r^\beta)<+\infty$.

3. Let D be a divisor on W and $f: C^n\to W$ a holomorphic map. Define

(6)
$$\delta(D)=1-\limsup_{r\to+\infty}[N(D,r)/T(D,r)].$$
$$\Theta(D)=1-\limsup_{r\to+\infty}[\overline{N}(D,r)/T(D,r)].$$
$$\theta(D)=\liminf_{r\to+\infty}[(N(D,r)-\overline{N}(D,r))/T(D,r)],$$
$$\gamma_1(D)=\liminf_{r\to+\infty}[N_1(r)/T(D,r)].$$

Remark. The quantity $\delta(D)$ is called the *defect* of D and $\theta(D)$ is called the *ramification index* of D. It is easily seen that

$$0 \leqq \delta(D) \leqq 1, \quad 0 \leqq \Theta(D) \leqq 1, \quad 0 \leqq \theta(D) \leqq 1, \quad \delta(D)+\theta(D) \leqq \Theta(D).$$

If $f(C^n) \cap D = \emptyset$, then $\delta(D)=1$, $\Theta(D)=1$, and $\theta(D)=0$.

Lemma 1. *Let $D_1, \cdots, D_k$ be divisors on W and let $D=D_1+\cdots+D_k$. Then we have*

$$\text{(i)} \quad \sum_{i=1}^{k} \left\{\liminf_{r\to+\infty} [T(D_i, r)/T(D, r)]\right\}\delta(D_i) \leqq \delta(D),$$

$$(7) \quad \text{(ii)} \quad \sum_{i=1}^{k} \left\{\liminf_{r\to+\infty} [T(D_i, r)/T(D, r)]\right\}\Theta(D_i) \leqq \Theta(D),$$

$$\text{(iii)} \quad \sum_{i=1}^{k} \left\{\liminf_{r\to+\infty} [T(D_i, r)/T(D, r)]\right\}\theta(D_i) \leqq \theta(D).$$

Proof. Clearly, given $\varepsilon>0$, the definition (6) implies

$$T(D_i, r)(\delta(D_i)-\varepsilon) < T(D_i, r) - N(D_i, r),$$

for sufficiently large r. Since $N(D, r)=N(D_1, r)+\cdots+N(D_k, r)$ and $T(D, r)=T(D_1, r)+\cdots+T(D_k, r)$, we have

$$\sum_{i=1}^{k} T(D_i, r)(\delta(D_i)-\varepsilon) < T(D, r) - N(D, r),$$

from which follows

$$\sum_{i=1}^{k} \left\{\liminf_{r\to+\infty} [T(D_i, r)/T(D, r)]\right\}(\delta(D_i)-\varepsilon) \leqq \delta(D).$$

Letting $\varepsilon \to 0$, we get the inequality (i). Noting that $\overline{N}(D, r) \leqq \overline{N}(D_1, r) + \cdots + \overline{N}(D_k, r)$, we can similarly show (ii), (iii).

Proposition 3. *Let $D=D_1+\cdots+D_k$ be a divisor on W saisfying the condition* (4). *Then*

$$(8) \qquad N(D, r) - \sum_{i=1}^{k} \overline{N}(D_i, r) \leqq N_1(r).$$

Proof. Set $S=\{$the singular locus of Supp $(f^*D)\}$. Take a point $x \in (\text{Supp}\,(f^*D)) - S$, and let $(z_1, \cdots, z_n)$ be holomorphic coordinates around x such that Supp $(f^*D)=\{z_1=0\}$ at x. By (4), we can take local coordinates $(w_1, \cdots, w_n)$ around $f(x)$ such that $D_i=\{w_i=0\}$, $i=1, \cdots, j$, $j \leqq k$, at $f(x)$. Writing f as

$$z=(z_1, \cdots, z_n) \to w_i = f_i(z), \qquad i=1, \cdots, n,$$

we have

$$\begin{cases} f_i(z) = z_1^{m_i} \cdot g_i(z), & g_i(x) \neq 0, \quad i=1, \cdots, j, \\ f_i(x) \neq 0, & \qquad\qquad\quad i=j+1, \cdots, n, \end{cases}$$

where each m_i is the multiplicity of f^*D_i at x. Hence

$$f^*D_i - \text{Supp}\,(f^*D_i) = \begin{cases} (m_i-1)\{z_1=0\}, & i=1, \cdots, j, \\ 0, & i=j+1, \cdots, k. \end{cases}$$

Moreover we see readily that

$$J_f = z_1^m \cdot G(z), \qquad m=\sum_{i=1}^{k}(m_i-1), \qquad (J_f) \geqq m\{z_1=0\}, \text{ at } x.$$

Thus we have

$$f^*D - \sum_{i=1}^{k} \text{Supp}\,(f^*D_i) \leqq (J_f), \qquad \text{at } x.$$

This holds outside S, and since $\text{codim}_{C^n} S \geqq 2$, this holds in C^n.

Q.E.D.

Remark. From (8), it follows that

$$\sum_{i=1}^{k} \left\{\liminf_{r\to+\infty} [T(D_i, r)/T(D, r)]\right\}\theta(D_i) \leqq \gamma_1(D).$$

In case $n=1$, we have $N(D,r)-\overline{N}(D,r)=N_1(r)$, which implies that $\theta(D)=\gamma_1(D)$.

Theorem 3 (Defect relations). *Let $D_1,\cdots,D_k$ be non-singular divisors on W such that $D=D_1+\cdots+D_k$ has only normal crossings. Assume that there exist rational numbers $q_0,\cdots,q_k$ such that*

$$\kappa(q_0K_W+\textstyle\sum_{i=1}^k q_iD_i, W)=n.$$

Let $f: \boldsymbol{C}^n\to W$ be a non-degenerate holomorphic map. Then

(9)
(i) $\delta(D)+\gamma_1(D)\leqq\limsup_{r\to+\infty}[-T(K_W,r)/T(D,r)]$,

(ii) $\sum_{i=1}^k\left\{\liminf_{r\to+\infty}[T(D_i,r)/T(D,r)]\right\}\Theta(D_i)\leqq\limsup_{r\to+\infty}[-T(K_W,r)/T(D,r)]$.

Proof. Letting $L=q_0K_W+q_1[D_1]+\cdots+q_k[D_k]$, we have, by (5),

$$T(D,r)-N(D,r)+N_1(r)\leqq -T(K_W,r)+O(\log T(L,r)),$$

for $r\notin E$. Dividing this by $T(D,r)$, we obtain

$$\delta(D)+\gamma_1(D)\leqq(-T(K_W,r)/T(D,r))+O((\log T(L,r))/T(D,r)). \tag{10}$$

On the other hand, in consequence of (2), given $\varepsilon>0$, letting r large enough, we may assume that $(\log T(L,r))/T(L,r)<\varepsilon$. Hence we get

$$\delta(D)+\gamma_1(D)\leqq(-T(K_W,r)/T(D,r))+(\varepsilon CT(L,r)/T(D,r)),$$

where C is a constant. Note that

$$T(L,r)/T(D,r)=q_0(T(K_W,r)/T(D,r))+\textstyle\sum_{i=1}^k q_i(T(D_i,r)/T(D,r)),$$
$$\leqq q_0(T(K_W,r)/T(D,r))+q,$$

where $q=q_0+\cdots+q_k$. Therefore

$$\delta(D)+\gamma_1(D)\leqq(1-\varepsilon Cq_0)(-T(K_W,r)/T(D,r))+\varepsilon Cq,$$

from which follows

$$\delta(D)+\gamma_1(D)\leqq(1-\varepsilon Cq_0)\left\{\limsup_{r\to+\infty}[-T(K_W,r)/T(D,r)]\right\}+\varepsilon Cq.$$

Taking the limit as $\varepsilon\to 0$, we obtain the inequality (i).

Combining (10) with (8), we get similarly

$$\textstyle\sum_{i=1}^k(T(D_i,r)/T(D,r))\Theta(D_i)\leqq(-T(K_W,r)/T(D,r))+(\varepsilon CT(L,r)/T(D,r)), \tag{11}$$

which proves the inequality (ii). Q.E.D.

Corollary. *If $\kappa(D,W)=n$, then the inequalities* (9) *hold.*

Proof. It suffices to put $q_0=0$, $q_i=1$, $i=1,\cdots,k$.

Corollary. *If $\kappa(K_W+D,W)=n$, then the inequalities* (9) *hold.*

Corollary (cf. [1], [3], [8]). *If $\kappa(K_W+D,W)=n$, then $f(\boldsymbol{C}^n)\cap D\neq\emptyset$.*

Example 1. Let $W=\boldsymbol{P}_n$ and let D_i be a hypersurface of degree d_i, respectively, for $i=1,\cdots,k$. Assume that $D=D_1+\cdots+D_k$ satisfies the condition (4). Let H be the hyperplane bundle. Since $K_{\boldsymbol{P}_n}=-(n+1)H$, $[D_i]=d_iH$, we get

$$-T(K_{\boldsymbol{P}_n},r)/T(D,r)=(n+1)/d,\quad T(D_i,r)/T(D,r)=d_i/d,\quad d=\textstyle\sum_{i=1}^k d_i.$$

Hence we obtain

$$\textstyle\sum_{i=1}^k d_i\delta(D_i)\leqq n+1,\qquad \sum_{i=1}^k d_i\Theta(D_i)\leqq n+1,\qquad \sum_{i=1}^k d_i\theta(D_i)\leqq n+1.$$

4. Let D be an irreducible divisor on W and $f: C^n \to W$ a holomorphic map such that $f(C^n) \not\subset D$. We write $f^*D = \sum_s m_s Z_s$, with Z_s irreducible. We say that f is *ramified over D with multiplicity at least e* if $m_s \geqq e$ holds for all s.

Lemma 2. *If f is ramified over D with multiplicity at least e, then we have*

$$\Theta(D) \geqq 1 - (1/e). \tag{12}$$

Proof. Since

$$f^*D = \sum_s m_s Z_s \geqq e(\sum_s Z_s) = e\,(\mathrm{Supp}\,(f^*D)),$$

we get

$$N(D, r) \geqq e\bar{N}(D, r).$$

Using this inequality and (1), we obtain

$$1 - (\bar{N}(D, r)/T(D, r)) \geqq 1 - (N(D, r)/eT(D, r)) \geqq 1 - (1/e).$$

Q.E.D.

Theorem 4 (Theorem 1 in [8])*). *Let $D_1, \cdots, D_k$ be non-singular divisors on W such that $D = D_1 + \cdots + D_k$ has only normal crossings. Let $f: C^n \to W$ be a non-degenerate holomorphic map which is ramified over D_i with multiplicity at least e_i, respectively. Then*

$$\kappa(K_W + \sum_{i=1}^k (1 - (1/e_i)) D_i, W) < n.$$

Proof. Let $L = K_W + (1 - (1/e_1))[D_1] + \cdots + (1 - (1/e_k))[D_k]$. Assume that $\kappa(L, W) = n$. Using (11) and (12), we have

$$T(L, r)/T(D, r) \leqq \varepsilon C T(L, r)/T(D, r),$$

for large $r \notin E$. From this and (3), it follows that

$$0 < (1 - \varepsilon C)(T(L, r)/T(D, r)) \leqq 0,$$

for sufficiently small ε and for large $r \notin E$. This is a contradiction.

Q.E.D.

Example 2. Let $D_1, \cdots, D_k$ be as in Example 1. Suppose that a non-degenerate holomorphic map $f: C^n \to P_n$ is ramified over each D_i with multiplicity at least e_i. Then

$$\sum_{i=1}^k d_i (1 - (1/e_i)) \leqq n + 1.$$

Remark. Shiffman [9] proved the second main theorem for meromorphic maps. So the results in this paper are valid for meromorphic maps. As for the case in which D has more general singularities, see [8], [9].

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*) Drouilhet [2] has obtained a similar result independently.

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