163. *Kummer Surfaces in Characteristic 2*

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§ 0. **Introduction.** Let A be an abelian surface (i.e. abelian variety of dim 2) defined over a field of characteristic p ($p=0$ or a prime number). Denoting by ι the inversion of A ($\iota(u)=-u, u\in A$), we consider the quotient surface A/ι, which has only isolated singularities corresponding to the points of order 2 of A. When $p\neq 2$, A/ι has 16 ordinary double points and by blowing up these points, we get a $K3$ surface (i.e. regular surface with a trivial canonical divisor), called the Kummer surface of A.

When $p=2$, the situation is a little different. The number of singular points of A/ι is smaller (4, 2 or 1), but they are more complicated singularities. In this note, we consider the case where $A=E\times E'$ is a product of elliptic curves, and instead of directly looking at the singularities of A/ι and their resolution, we study the non-singular elliptic surface (Kodaira-Néron model) of the fibration $A/\iota\to E/\iota=\boldsymbol{P}^1$, induced by the projection $A\to E$. We define the *Kummer surface* of A, $Km(A)$, to be this non-singular elliptic surface, birationally equivalent to A/ι. Rather unexpectedly, we have

Proposition 1. *Assume $p=2$ and let $A=E\times E'$. Then*

(i) *$Km(A)$ (and hence A/ι) is a rational surface, if E and E' are supersingular elliptic curves.*

(ii) *$Km(A)$ is a $K3$ surface in all other cases.*

Proposition 2. *The Picard number ρ of $Km(A)$ in the case* (ii) *is given as follows:*

$$\rho=\begin{cases}18 & \text{if } E\not\sim E',\\ 19 & \text{if } E\sim E',\ End(E)=\boldsymbol{Z},\\ 20 & \text{if } E\sim E',\ End(E)\neq\boldsymbol{Z}.\end{cases}$$

Here "$\sim$" indicates isogeny. Note in particular that the $K3$ surfaces $Km(A)$ in (ii) cannot be supersingular in the sense of M. Artin [1], nor unirational (cf. [9]). It will be interesting to study the singularities of A/ι and to obtain its non-singular model for any abelian surface (or variety) in characteristic 2. For example, we can ask: (i) Is A/ι rational if A has no point of exact order 2? (In this case, A/ι is unirational.) (ii) Is A/ι birationally equivalent to a $K3$ surface if A has at least one point of exact order 2? We shall consider these questions in some occasion.

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§ 1. Elliptic curves in characteristic 2. We fix an algebraically closed field k of characteristic $p=2$. For each $j \in k$, we denote by E_j an elliptic curve with the absolute invariant j. Explicitly E_j can be defined by the equation (cf. [3]):

$$y^2+axy+cy=x^3+bx, \tag{1}$$

where

$$\begin{cases} a=b=0, \quad c=1, & \text{if } j=0 \\ a=j^{-1/6}, \quad b=aj^{-1}, \quad c=0 & \text{if } j\neq 0. \end{cases} \tag{2}$$

We choose the unique point at infinity as the origin of the group law on E_j. The inversion ι of E_j is then expressed by

$$(x, y)\to(x, y+ax+c). \tag{3}$$

It follows that E_j is supersingular (i.e. no point of exact order $p=2$) if and only if $j=0$. Note also that $(x, y)\to x$ induces the isomorphism $E_j/\iota\simeq \boldsymbol{P}^1$.

§ 2. Kummer surfaces. Let us consider the abelian surface

$$A=E_j\times E_{j'}, \tag{4}$$

in which $E_{j'}$ is defined by the equation (1) with a, b, c replaced by a', b', c'. Denoting the coordinates of the first and second factor of A by (x, y) and (x', y'), we identify the function field $k(A)$ of A with $k(x, y, x', y')$. The function field of A/ι is isomorphic to the subfield of $k(A)$ of those elements invariant under

$$(x, y, x', y')\to(x, y+ax+c, x', y'+a'x'+c'). \tag{5}$$

Putting

$$z=(ax+c)y'-(a'x'+c')y, \tag{6}$$

we have

$$k(A/\iota)=k(x, x', z) \tag{7}$$

with the relation

$$z^2-(ax+c)(a'x'+c')z=(ax+c)^2(x'^3+b'x')-(a'x'+c')^2(x^3+bx). \tag{8}$$

Let $f_1 : A/\iota\to E_j/\iota=\boldsymbol{P}^1$ be the morphism induced by the projection $A\to E_j$. We put

$$\Sigma=\begin{cases}\{\infty\} & (j=0) \\ \{\infty, 0\} & (j\neq 0).\end{cases} \tag{9}$$

For each $x\in \boldsymbol{P}^1-\Sigma$, the fibre $f_1^{-1}(x)$ is an elliptic curve, defined by the equation (8) over the field $k(x)$. By the method of Kodaira-Néron ([4], [5]), we can replace the fibre $f_1^{-1}(v)$ for $v\in\Sigma$ by a suitable configuration of curves C_v so that we obtain a non-singular elliptic surface $f : X\to \boldsymbol{P}^1$ with $f^{-1}(v)=C_v$. The type of singular fibres C_v, which depends on $\{j, j'\}$, will be explicitly given in § 4. This surface X will be called the *Kummer surface*, $Km(A)$, of $A=E_j\times E_{j'}$ (cf. § 6). Note that $f : X\to\boldsymbol{P}^1$ admits a section; in fact, the map $x\to(x, o')$ of E_j into A induces such a section (o'=the origin of $E_{j'}$).

§ 3. **Generalities on elliptic surfaces.** We recall here some facts about elliptic surfaces (in any char. p). Let $f: X \to \boldsymbol{P}^1$ be a non-singular elliptic surface over $\boldsymbol{P}^1$ such that

(i) no fibre contains an exceptional curve of the first kind,

(ii) f admits a section, and

(iii) the set $\Sigma = \{v \in \boldsymbol{P}^1 \mid f^{-1}(v) \neq \text{elliptic curve}\}$ is non-empty.

For $v \in \Sigma$, let m_v denote the number of irreducible components in the fibre $f^{-1}(v)$, and let ord_v denote the order of the discriminant of the minimal Weierstrass equation at v (cf. [6]). Moreover let c_2 (or ρ) be the Euler number (or the Picard number) of X. Then we have

(10) $$c_2 = \sum_v \mathrm{ord}_v \qquad \text{(cf. [4], [6])}$$

(11) $$\rho = r + 2 + \sum_v (m_v - 1) \qquad \text{(cf. [6], [7])},$$

where r is the rank of group of rational points of the generic fibre. Furthermore we have the following criteria:

(12) $$X: \text{rational} \Longleftrightarrow c_2 = 12,$$

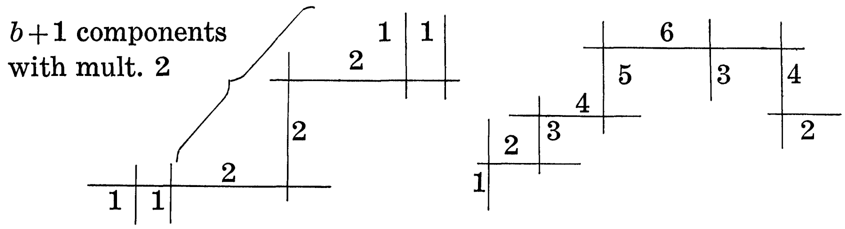
(13) $$X: K3 \Longleftrightarrow c_2 = 24.$$

The proof of (12), (13) depends on the fact that the canonical divisor of X is induced by a certain divisor of $\boldsymbol{P}^1$ of degree $-2 + c_2/12$ (cf. [4], [2]). In addition, we use the Castelnuovo's criterion of rationality for (12).

§ 4. **Singular fibres of Kummer surfaces.** We go back to the situation in § 2 ($p=2$). Let $X = Km(A)$ be the Kummer surface of $A = E_j \times E_{j'}$, together with the morphism $f: X \to \boldsymbol{P}^1$ coming from the projection $A \to E_j$. The number and type of singular fibres C_v ($v \in \Sigma$) are summarized in the following table. (The last line of the table for $p \neq 2$ is added for the sake of comparison.)

	$\lvert\Sigma\rvert$	Type in [4] and [5]		m_v	ord_v
$j=0,\ j'=0$	1	I_0^*	$c4$	5	12
$j=0,\ j'\neq 0$	1	I_{12}^*	$c5_{12}$	17	24
$j\neq 0,\ j'=0$	2	II^*	$c8$	9	12
$j\neq 0,\ j'\neq 0$	2	I_4^*	$c5_4$	9	12
any product in char. $p\neq 2$	4	I_0^*	$c4$	5	6

I_b^* ($b \geqslant 0$) $\qquad$ II^*



In order to verify this table, we rewrite in each case the equation (8) over $k(x)$ into a suitable Weierstrass form, and compare with the classification of singular fibres in [5]. We omit the computation.

§ 5. **Proof of Propositions 1, 2.** From the above table, we have

$$\sum_v \mathrm{ord}_v = \begin{cases} 12 & \text{if } j=j'=0 \\ 24 & \text{otherwise.} \end{cases} \tag{14}$$

This proves Proposition 1 in view of the criteria (12), (13). For the part (i), i.e. for the rationality of A/ι with $A=E_0\times E_0$, we can also give a simple direct proof, avoiding such a deep criterion as (12). In fact, the equation (8) for the case $j=j'=0$ reads

$$z^2-z=x^3-x'^3. \tag{15}$$

Putting $x'=x+s$, we have

$$z^2-z+s(x^2+sx+s^2)=0. \tag{16}$$

Regarded as a quadratic equation in z and x with coefficients in the field $k(s)$, (16) has a rational point $(z, x)=(0, \omega s)$ where ω is a primitive cubic root of 1. If we put $t=z/(x-\omega s)$, (16) is rewritten as

$$(t^2+s)x=t+\omega t^2 s+\omega^2 s^2.$$

Hence we have by (7)

$$k(A/\iota)=k(x, x', z)=k(s, t), \tag{17}$$

which shows the rationality of A/ι when $j=j'=0$.

For the Picard number $\rho(X)$ of $X=Km(A)$, $A=E_j\times E_{j'}$, we have (cf. [8])

$$\begin{aligned} \rho(X)&=\rho(A/\iota)+\sum_v (m_v-1) \\ &=\text{rank Hom}\,(E_j, E_{j'})+2+\sum_v (m_v-1). \end{aligned} \tag{18}$$

Therefore Proposition 2 follows immediately from the table of § 4. By the way, comparing (18) with (11), we see that the rank r in (11) is also given by

$$r=\text{rank Hom}\,(E_j, E_{j'}). \tag{19}$$

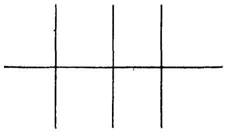
§ 6. **Remarks.** (a) Let $A=E_j\times E_{j'}$ with $(j, j')\neq(0, 0)$. Suppose we have another decomposition of A:

$$A\simeq E_{j_1}\times E_{j'_1} \qquad (\text{e.g. } j_1=j',\ j'_1=j). \tag{20}$$

Let X (resp. X_1) denote the non-singular elliptic surface over $\boldsymbol{P}^1$ associated with $A/\iota\to E_j/\iota$ (resp. $A/\iota\to E_{j_1}/\iota$). Then both X and X_1 are $K3$ surfaces by Proposition 1 (ii), birationally equivalent to A/ι. By the minimality of a $K3$ surface, X and X_1 are naturally isomorphic. Thus the definition (§ 2) of the Kummer surface of A does not depend on the way how A decomposes as a product of elliptic curves.

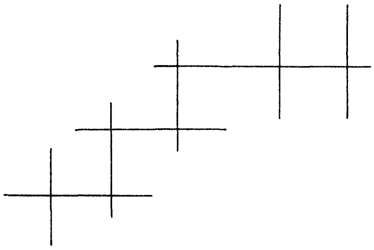
(b) The results in § 4 on singular fibres suggest that each singular point of A/ι for $A=E_j\times E_{j'}$ (j, j' not both zero) has the minimal resolution consisting of the following configuration of non-singular rational curves:

D_4



$jj' \neq 0$

D_8



$jj' = 0$

This latter fact has recently been shown by M. Artin [10].

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