158. *Fundamental Solution of Partial Differential Operators of Schrödinger's Type. II*

The Space-Time Approach

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§ 1. **Introduction.** In the previous note [2] we constructed the fundamental solution of $i\nu\dfrac{\partial}{\partial t}+1/2\Delta$, where Δ is the Laplace operator associated with a Riemannian metric $ds^2=\sum_{ij} g_{ij}(x)dx_i dx_j$ in R^n satisfying some conditions. There we made use of discussions of classical orbits in the phase space. In this note discussing in the spacetime, we shall construct the fundamental solution of $\nu i\dfrac{\partial}{\partial t}+\Delta$, $\nu>0$. This will be closer to the original Feynman's idea [1]. Assumptions will be found in § 2 and results will be found in § 4. In § 3 we shall construct parametrix. The outline of proof will be given in § 5. The main Lemma proof of which is too long to be presented in this short note will be proved in the subsequent paper [3].

§ 2. **Assumptions.** Let $|x-y|$ be the Euclidean distance from y to x and $r(x, y)$ be the geodesic distance from y to x. Our assumptions are the following ones:

(A–I) for any two points x, y in R^n, there exists unique geodesic joining x to y.

(A–II) the metric ds^2 coincides with the Euclidean metric outside compact set K.

(A–III) there exists a constant $C>0$ such that

$$|\mathrm{grad}_x\,(r^2(x, y)-r^2(x, z))|\geqq C\,|y-z|. \tag{1}$$

(A–IV) for any multi-indices α with $|\alpha|\geqq 2$, there exists a constant $C>0$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}(r^2(x, y)-r^2(x, z))\right|\leqq C\,|y-z|. \tag{2}$$

§ 3. **Parametrix.** We make use of the parametrix of the form

$$E_N(t, x, y)=(\nu/4\pi ti)^{1/2n}\exp\,(i\nu r^2(x, y)/4t)e(t, x, y), \tag{3}$$

$$e(t, x, y)=\sum_{j=0}^{N}(it/\nu)^j e_j(x, y). \tag{4}$$

If we use geodesic polar coordinates with center at y, the function

$e_j(x, y)$ is determined in the following manner;

$$r\frac{d}{dr}e_{j+1}+\left(\frac{1}{2}r\frac{d}{dr}\log\sqrt{g}+j+1\right)e_{j+1}=\Delta e_j, \tag{5}$$

$$e_{-1}=0, \qquad e_0(y, y)=1.$$

The solution of these equations are

$$e_0(x, y)=(g(x)/(g(y))^{1/4}, \tag{6}$$

$$e_j(x, y)=e_0(x, y)r(x, y)^{-j}\int_0^{r(x,y)}\frac{r(z, y)^{j-1}}{e_0(z, y)}\Delta_z e_{j-1}(z, y)dr(z, y). \tag{7}$$

Integral is taken along the geodesic joining x to y. From this construction we have

$$\begin{aligned}&\left(\nu i\frac{\partial}{\partial t}+\Delta\right)E_N(t, x, y)\\ &\quad=-(\nu/4\pi it)^{1/2n}(it/\nu)^N \exp(i\nu r^2(x, y)/4t)\Delta e_N(x, y).\end{aligned} \tag{8}$$

Since $e_0(x, y)=g(y)^{-1/4}$ if $x\notin K$ and $=g(x)^{1/4}$ if $y\notin K$, we have, for any multi-indices α, β,

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial y}\right)^{\beta}e_j(x, y)\right|\leqq C \tag{9}$$

with $j=0$. Making use of (7) we see easily that (9) holds for any $j=0, 1, \cdots, N$.

§ 4. **Results.** Let us define an integral transformation $E_N(t)$ by

$$E_N(t)f(x)=\int_{R^n}E_N(t, x, y)f(y)\sqrt{g(y)}dy. \tag{10}$$

Then our results are the following theorems.

Theorem I. *$E_N(t)$ is a bounded linear transformation in $L^2(R^n, \sqrt{g}dx)$.*

Theorem II (cf. Feyman [1]).

$$\lim_{k\to\infty}\|E_N(t/k)E_N(t/k)\cdots E_N(t/k)-\exp(i\nu^{-1}t\Delta)\|=0, \tag{11}$$

where $\|\ \|$ is the operator norm in $L^2(R^n, \sqrt{g}dx)$ and $\exp(i\nu^{-1}t\Delta)$ is the one parameter group of unitary operators whose generator is $i\nu^{-1}\Delta$. (Cf. Stone [4].)

§ 5. **Outline of the proof.** We introduce another linear integral transformation $F(t)$ as the following;

$$\begin{aligned}&F(t)f(x)\\ &\quad=(\nu/4\pi it)^{1/2n}(t/\nu i)^N\int_{R^n}\Delta e_N(x, y)\exp(i\nu r^2(x, y)/4t)f(y)\sqrt{g(y)}dy.\end{aligned} \tag{12}$$

Our fundamental lemma is

Lemma. *Let $a(x, y)$ be a function in $C^\infty(R^n\times R^n)$ which satisfies the same estimate as* (9). *Set*

$$Af(x)=\int_{R^n}a(x, y)\exp(i\lambda r^2(x, y))f(y)dy, \qquad \lambda>0. \tag{13}$$

Then there exists a constant $C>0$ independent of λ and f such that we have

$$\|Af\| \leqq C\lambda^{-1/2n}\|f\| \tag{14}$$

for any f in $C_0^\infty(R^n)$. *Here* $\|\ \|$ *is the norm in* $L^2(R^n, \sqrt{g}dx)$.

Theorem I is an immediate consequence of Lemma. We again apply this lemma and obtain estimates of the norm of the operator $F_N(t)$, that is,

$$\|F_N(t)\| \leqq C|t/\nu|^N. \tag{15}$$

We denote $U(t) = \exp(i\nu^{-1}t\Delta)$. Then the difference $R(t) = E_N(t) - U(t)$ can be written as

$$R(t) = \int_0^t U(t-s)F_N(s)ds. \tag{16}$$

The norm of it is majorized as

$$\|R(t)\| \leqq C|t/\nu|^N|t|. \tag{17}$$

The k-products of $E_N(t/k)$ turns out to be

$$\begin{aligned} &E_N(t/k)E_N(t/k)\cdots E_N(t/k) \\ &\quad = (U(t/k) - R(t/k))\cdots(U(t/k) - R(t/k)). \end{aligned} \tag{18}$$

Since $U(t/k)$ is unitary, we obtain

$$\begin{aligned} \|E_N(t/k)E_N(t/k)\cdots E_N(t/k) - U(t)\| &\leqq \sum_{j=1}^{k}\binom{k}{j}\|R(t/k)\|^j \\ &= (1 + \|R(t/k)\|)^k - 1. \end{aligned} \tag{19}$$

This tends to 0 as k goes to ∞ if $N \geqq 1$. Our theorems have been proved up to the proof of our lemma which will be given in the subsequent note [3].

References

[1] R. Feynman: Space-time approach to non-relativistic quantum mechanics. Review of Modern Physics, **20**, 367–384 (1948).

[2] D. Fujiwara: Fundamental solution of partial differential operators of Schrödinger's type. I. Proc. Japan Acad., **50**, 566–569 (1974).

[3] ——: On the boundedness of integral transformations with highly oscillatory kernels (to appear).

[4] M. H. Stone: Linear Transformations in Hilbert Space and Their Applications to Analysis. Amer. Math. Soc., New York (1932).