157. On the Trotter-Lie Product Formula*)

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1. In [1, Proposition 7.9] Chernoff gives an example of a pair A, B of nonnegative selfadjoint operators such that

(1) $$(e^{-tA/n}e^{-tB/n})^n \xrightarrow[s]{} 0 \quad \text{as } n\to\infty,\ t>0,$$

where $\xrightarrow[s]{}$ denotes strong convergence. In this example, A is a differential operator of common type while B is an operator of multiplication with a highly singular function; the proof makes essential use of the Wiener integral.

In what follows we shall show that if A, B are nonnegative selfadjoint, (1) is true whenever $\boldsymbol{D}(A^{1/2})\cap\boldsymbol{D}(B^{1/2})=\{0\}$, which is the case in Chernoff's example. [$\boldsymbol{D}(T)$ denotes the domain of T.] Furthermore, we shall show that (1) is true in the general case if applied to a vector orthogonal to $\boldsymbol{D}(A^{1/2})\cap\boldsymbol{D}(B^{1/2})$.

We shall consider this problem for a more general sequence

(2) $$U_n(t)=[f(tA/n)g(tB/n)]^n, \qquad n=1,2,\cdots,$$

where f, g are taken from the class of real-valued, Borel measurable functions ϕ on $[0,\infty)$ such that

(3) $$0<\phi(t)\le 1, \quad \phi(0)=1, \quad \phi'(0)=-1.$$

$\phi(t)=e^{-t}$ belongs to this class. Another example is $\phi(t)=(1+t)^{-1}$, which is perhaps more important in connection with approximation theory in differential equations.

We note that (3) already implies that

(4) $$\phi(tA)\xrightarrow[s]{} 1, \qquad t\downarrow 0,$$

whenever A is nonnegative selfadjoint.

To prove our results, we need a mild additional condition for at least one of f and g, namely

(5) $t^{-1}[1-\phi(t)]$ is monotone nonincreasing on $0<t<\infty$.

Note that (5) is again satisfied by $\phi(t)=e^{-t}$ and $(1+t)^{-1}$.

We can now state our main theorem.

Theorem 1. *Let A, B be nonnegative selfadjoint operators in a Hilbert space $\boldsymbol{H}$. Assume that both f and g satisfy* (3) *and at least one of them satisfies* (5). *If $v\in\boldsymbol{H}$ is orthogonal to $\boldsymbol{D}(A^{1/2})\cap\boldsymbol{D}(B^{1/2})$, then $U_n(t)v\to 0$ as $n\to\infty$, uniformly on compact sets of $t>0$.*

Theorem 1 raises the question as to what happens to $U_n(t)v$ if

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$v \in \boldsymbol{H}' \equiv$ closure of $\boldsymbol{D}(A^{1/2}) \cap \boldsymbol{D}(B^{1/2})$. It is natural to expect that $U_n(t)v \to e^{-tC'}v$, where $C' = A \dotplus B$ (*the form-sum*); C' is by definition the selfadjoint operator in $\boldsymbol{H}'$ associated with the nonnegative closed quadratic form $u \mapsto \|A^{1/2}u\|^2 + \|B^{1/2}u\|^2$, which is densely defined in $\boldsymbol{H}'$. We shall show that this is indeed the case under a slightly stronger assumption on one of f and g, namely

(6) $t^{-1}[\phi(t)^{-1} - 1]$ is monotone nonincreasing on $0 < t < \infty$.

Note that $\phi(t) = (1+t)^{-1}$ satisfies (6) but e^{-t} *does not*, unfortunately.

Theorem 2. *Let A, B be as in Theorem* 1 *and let $\boldsymbol{H}'$ be the closure of $\boldsymbol{D}(A^{1/2}) \cap \boldsymbol{D}(B^{1/2})$. Assume that both f and g satisfy* (3) *and one of them satisfies* (5) *while the other satisfies* (6). *Then*

(7) $$U_n(t) \xrightarrow[s]{} e^{-tC'} \oplus 0 \qquad \text{as } n \to \infty,$$

the direct sum corresponding to $\boldsymbol{H} = \boldsymbol{H}' \oplus \boldsymbol{H}'^{\perp}$, where $C' = A \dotplus B$ (see above). The convergence in (7) *is uniform on compact sets of $t > 0$, and on compact sets of $t \geq 0$ when applied to $u \in \boldsymbol{H}'$.*

Examples. Theorem 1 is applicable to the cases

$$U_n(t) = (e^{-tA/n}e^{-tB/n})^n, \qquad [(1+tA/n)^{-1}e^{-tB/n}]^n,$$
$$[e^{-tA/n}(1+tB/n)^{-1}]^n, \qquad [(1+tA/n)^{-1}(1+tB/n)^{-1}]^n.$$

Theorem 2 applies to all these cases except the first one, for which the question is open whether or not (7) is true.

In what follows we shall give a complete proof of Theorem 1, with a sketch of the proof of Theorem 2.

2. We shall prove Theorem 1 in a series of lemmas.

Lemma 1. *Let H_n, $n = 1, 2, \cdots$, be selfadjoint operators in $\boldsymbol{H}$ such that $1 \leq H_1 \leq H_2 \leq \cdots$ in the usual sense (see e.g.* [2, p. 330]). *Let $\boldsymbol{D}_0$ be the set of all $u = \bigcap_n \boldsymbol{D}(H_n^{1/2})$ such that $\sup_n \|H_n^{1/2}u\|$ is finite. Then $H_n^{-1}v \to 0$, $n \to \infty$, for all $v \perp \boldsymbol{D}_0$.*

Proof. Since $\{H_n^{-1}\}$ is a monotone nonincreasing sequence of bounded, nonnegative, selfadjoint operators, there is an operator K of the same kind such that $H_n^{-1} \xrightarrow[s]{} K$ (see e.g. [2, p. 452]).

Since $0 \leq K \leq H_n^{-1}$, we have $\|K^{1/2}u\| \leq \|H_n^{-1/2}u\|$ for $u \in \boldsymbol{H}$. A standard argument then shows that $K^{1/2}$ has range in $\boldsymbol{D}(H_n^{1/2})$ with $\|H_n^{1/2}K^{1/2}u\| \leq \|u\|$. Since this is true for all n, we have $K^{1/2}u \in \boldsymbol{D}_0$. Thus $K^{1/2}$ has range in $\boldsymbol{D}_0$, and so does K. Since K is selfadjoint, it follows that $Kv = 0$ for $v \perp \boldsymbol{D}_0$. Since $H_n^{-1} \xrightarrow[s]{} K$, this proves the lemma.

3. To continue with the proof, we may assume that f satisfies (5). (The other case can easily be reduced to this case, see the argument in [1, p. 48].) Set

(8) $$G(t) = t^{-1}[g(tB)^{-1} - f(tA)] \geq 0, \qquad t > 0.$$

$G(t)$ is in general unbounded, but it is selfadjoint with $\boldsymbol{D}(G(t)) = \boldsymbol{D}(g(tB)^{-1})$ because $f(tA)$ is bounded selfadjoint.

Lemma 2. $[1 + G(t)]^{-1}v \to 0$, $t \downarrow 0$, *for $v \perp \boldsymbol{D}' \equiv \boldsymbol{D}(A^{1/2}) \cap \boldsymbol{D}(B^{1/2})$.*

Proof. Since g satisfies (3), we can find a monotone nonincreasing function p on $[0, \infty)$ such that

(9) $$0 \le p(t) \le t^{-1}[g(t)^{-1}-1], \quad p(t) \uparrow 1 = p(0) \qquad \text{as } t \downarrow 0.$$

For example, it suffices to set $p(0)=1$ and

$$p(t) = \min\left\{1, \inf_{0<s\le t} s^{-1}[g(s)^{-1}-1]\right\}, \qquad t>0.$$

(9) implies that $t^{-1}[g(t\beta)^{-1}-1] \ge \beta p(t\beta)$ for $t>0$ and $\beta \ge 0$. Hence $t^{-1}[g(tB)^{-1}-1] \ge Bp(tB)$, so that

(10) $$G(t) \ge H(t) \equiv Bp(tB) + t^{-1}[1-f(tA)] \ge 0.$$

Thus the desired result follows from the following lemma.

Lemma 3. $[1+H(t)]^{-1}v \to 0$, $t \downarrow 0$, $v \perp \boldsymbol{D}'$.

Proof. $H(t)$ is again in general unbounded but nonnegative selfadjoint with $\boldsymbol{D}(H(t)) = \boldsymbol{D}(Bp(tB))$. An important property of $H(t)$ is that it is a monotone nondecreasing family as $t \downarrow 0$, since

(11) $$\beta p(t\beta) \uparrow \beta, \quad t^{-1}[1-f(t\beta)] \uparrow \beta \qquad \text{as } t \downarrow 0,$$

for any $\beta \ge 0$ (recall that f satisfies (5)).

Thus we can apply Lemma 1 with the sequence H_n replaced by the family $1+H(t)$ with $t \downarrow 0$. The desired result will follow if we show that $\|H(t)^{1/2}u\|$ is bounded as $t \downarrow 0$ if and only if $u \in \boldsymbol{D}'$. But

(12) $$\|H(t)^{1/2}u\|^2 = \|B^{1/2}p(tB)^{1/2}u\|^2 + (t^{-1}[1-f(tA)]u, u)$$

with $\boldsymbol{D}(H(t)^{1/2}) = \boldsymbol{D}(B^{1/2}p(tB)^{1/2})$. (12) is bounded as $t \downarrow 0$ if and only if each of the two terms on the right is bounded. In view of (11), it is easily seen from the spectral theorem and the monotone convergence theorem that the first term is bounded if and only if $u \in \boldsymbol{D}(B^{1/2})$ and the second term if and only if $u \in \boldsymbol{D}(A^{1/2})$. Thus (12) is bounded as $t \downarrow 0$ if and only if $u \in \boldsymbol{D}'$, as we wished to show.

4. Now we introduce the following operators.

(13) $$F(t) = f(tA)^{1/2} g(tB) f(tA)^{1/2}, \qquad t \ge 0,$$

(14) $$S(t) = t^{-1}[1-F(t)], \qquad t>0.$$

Note that $F(t)$ and $S(t)$ are bounded, nonnegative selfadjoint operators, with $F(t) \le 1$.

Lemma 4. *$[\lambda+S(t)]^{-1}v \to 0$, $t \downarrow 0$, $v \perp \boldsymbol{D}'$, uniformly on compact sets of $\lambda>0$.*

Proof. Since $S(t) \ge 0$, if suffices to prove the result for $\lambda=1$ (see [2, p. 427]). A simple computation gives

$$[1+S(t)]f(tA)^{1/2} \supset f(tA)^{1/2}[1+g(tB)G(t)]$$

with $G(t)$ given by (8). Hence

(15) $$[1+S(t)]f(tA)^{1/2}[1+G(t)]^{-1}v = f(tA)^{1/2}[1+g(tB)G(t)][1+G(t)]^{-1}v$$

for any $v \in \boldsymbol{H}$.

Assume now that $v \perp \boldsymbol{D}'$. Then Lemma 2 shows that

(16) $$[1+G(t)]^{-1}v \to 0, \qquad \text{hence} \quad G(t)[1+G(t)]^{-1}v \to v, \ t \downarrow 0.$$

Thus the right member of (15) tends to v; recall that $f(tA)^{1/2}$ and $g(tB)$ tend strongly to the identity by (4). Hence

(17) $$[1+S(t)]f(tA)^{1/2}[1+G(t)]^{-1}v-v\to 0, \qquad t\downarrow 0.$$

Since $\|[1+S(t)]^{-1}\|\le 1$, we can multiply (17) from the left by $[1+S(t)]^{-1}$, obtaining

$$f(tA)^{1/2}[1+G(t)]^{-1}v-[1+S(t)]^{-1}v\to 0, \qquad t\downarrow 0.$$

But the first term on the left tends to zero by (16) and $f(tA)^{1/2}\underset{s}{\to}1$. This proves the lemma.

5. We can now finish the proof of Theorem 1. We have

$$0\le F(t)=1-tS(t)\le[1+tS(t)]^{-1}, \qquad t>0,$$

by (14), since $S(t)$ is nonnegative selfadjoint. Noting that $F(t)$ and $S(t)$ commute, we obtain

$$0\le F(t/n)^{2n}\le[1+(t/n)S(t/n)]^{-2n}\le[1+2tS(t/n)]^{-1}$$
$$=(2t)^{-1}[(2t)^{-1}+S(t/n)]^{-1}, \qquad t>0.$$

Hence for $v\perp \boldsymbol{D}'$,

$$\|F(t/n)^n v\|^2\le(2t)^{-1}([(2t)^{-1}+S(t/n)]^{-1}v,v)\to 0, \qquad n\to\infty,$$

by Lemma 4, uniformly on compact sets of $t>0$. Thus we have proved that $F(t/n)^n v\to 0$. From this it is easy to deduce the desired result $U_n(t)v\to 0$ (see again [1, p. 48]).

6. Sketch of the proof of Theorem 2. We may assume that f satisfies (5) and g satisfies (6). Since these assumptions are stronger than in Theorem 1, we have $U_n(t)v\to 0$ for $v\in \boldsymbol{H}'^{\perp}$. Thus it only remains to show that

(18) $$U_n(t)u\to e^{-tC'}u, \quad n\to\infty, \quad u\in \boldsymbol{H}'.$$

To prove (18), it again suffices to show that

(19) $$F(t/n)^n u\to e^{-tC'}u, \quad n\to\infty, \quad u\in \boldsymbol{H}',$$

where F is given by (13).

According to a theorem due to Chernoff [1, Theorem 1.1], (19) is true if

(20) $$[1+S(t)]^{-1}u\to(1+C')^{-1}u, \quad t\downarrow 0, \quad u\in \boldsymbol{H}'.$$

It should be noted that Chernoff's theorem is originally for the case in which C' is the generator of a C_0-semigroup on $\boldsymbol{H}$, but the proof applies *mutatis mutandis* to the present case, in which C' is such a generator in $\boldsymbol{H}'$.

The argument in the proof of Lemma 4 then shows that (20) follows from

(21) $$[1+G(t)]^{-1}u\to(1+C')^{-1}u, \quad t\downarrow 0, \quad u\in \boldsymbol{H}'.$$

To prove (21) we use the assumption that g satisfies (6), which implies that we may choose $p(t)=t^{-1}[g(t)^{-1}-1]$ in the proof of Lemma 2, so that $G(t)=H(t)$. Since $H(t)$ is a nondecreasing family as $t\downarrow 0$, it is not difficult to establish (21) using a lemma that strengthens Lemma 1.

The writer plans in a future publication to discuss such a lemma in more detail, with applications in other directions.

References

[1] P. R. Chernoff: Product Formulas, Nonlinear Semigroups, and Addition of Unbounded Operators. Mem. Amer. Math. Soc. No. 140 (1974).
[2] T. Kato: Perturbation Theory for Linear Operators. Springer (1966).