155. *Note on Pure Subsystems*

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1. By a *right S-system* M_S over a semigroup S we mean a set M together with a mapping $(x, a) \to xa$ of $M \times S$ into M satisfying

$$x(ab) = (xa)b$$

for all $x \in M$ and $a, b \in S$. A non-empty subset N of a right S-system M_S is called an *S-subsystem* of M_S if $NS \subseteq N$. An S-subsystem N of a right S-system M_S is called *R-pure* in S if

$$N \cap Ma = Na$$

for all $a \in S$. Since the inclusion $\supseteq$ is true for every S-subsystem N of M_S, the essential requirement is

$$N \cap Ma \subseteq Na$$

for all $a \in S$. A right S-system M_S is called *R^*-pure* if every S-subsystem of M_S is R-pure in S.

In [3] the author proved that for a semigroup S with an identity the following conditions are equivalent:

(1) S is regular.

(2) Every unital right S-system M_S is R^*-pure.

(3) S is R^*-pure.

In this note we shall give another properties of pure S-subsystems of a right S-system M_S over a semigroup S. For the terminology not defined here we refer to the book by A. H. Clifford and G. B. Preston [1].

2. A subsemigroup B of a semigroup S is called a *bi-ideal* of S if $BSB \subseteq B$. We denote by $[b]$ the principal bi-ideal of a semigroup S generated by b in S, that is,

$$[b] = b \cup b^2 \cup bSb.$$

First we give the following.

Theorem 1. *For an S-subsystem N of a right S-system M_S over a semigroup S the following conditions are equivalent:*

(1) *N is R-pure in S.*

(2) $N \cap MB = NB$ *for all bi-ideals B of S.*

(3) $N \cap M[b] = N[b]$ *for all $b \in S$.*

Proof. First we assume that N is R-pure in S. Let B be any bi-ideal of S and $p = qb$ $(p \in N, q \in M, b \in B)$ any element of $N \cap MB$. Then we have

$$p = qb \in N \cap Mb = Nb \subseteq NB$$

and so we have

$$N \cap MB \subseteq NB.$$

Since the converse inclusion always holds, we have

$$N \cap MB = NB$$

for all bi-ideals B of S. Therefore we obtain that (1) implies (2). It is clear that (2) implies (3). We assume that (3) holds. Let a be any element of S and $p=qa\ (p \in N, q \in M)$ any element of $N \cap Ma$. Then we have

$$\begin{aligned} p=qa \in N \cap M[a] &= N[a] \\ &= N(a \cup a^2 \cup aSa) \\ &= Na \cup Na^2 \cup N(aSa) \\ &= Na \cup (Na)a \cup (NaS)a \\ &\subseteq Na \cup Na \cup Na \\ &= Na \end{aligned}$$

and so we have

$$N \cap Ma \subseteq Na$$

for all $a \in S$. Thus we obtain that N is R-pure in S and that (3) implies (1). This completes the proof of the theorem.

A right ideal A of a semigroup S is called *R-pure* in S if

$$A \cap Sa = Aa$$

for all $a \in S$. A semigroup S is called *R^*-pure* if every right ideal of S is R-pure in S.

Since any right ideal of a semigroup S is an S-subsystem of a right S-system S_S, the following corollary is immediate from the above theorem.

Corollary 2. *For a right ideal A of a semigroup S the following conditions are equivalent*:

(1) *A is R-pure in S.*

(2) $A \cap SB = AB$ *for all bi-ideals B of S.*

(3) $A \cap S[b] = A[b]$ *for all $b \in S$.*

We denote by $[a]_r$ the principal right ideal of a semigroup S generated by a in S, that is,

$$[a]_r = a \cup aS.$$

Corollary 3. *For a semigroup S the following conditions are equivalent*:

(1) *S is R^*-pure.*

(2) $A \cap SB = AB$ *for all right ideals A and for all bi-ideals B of S.*

(3) $A \cap S[b] = A[b]$ *for all right ideals A and for all $b \in S$.*

(4) $[a]_r \cap Sb = [a]_r b$ *for all $a, b \in S$.*

(5) $[a]_r \cap SB = [a]_r B$ *for all $a \in S$ and for all bi-ideals B of S.*

(6) $[a]_r \cap S[b] = [a]_r[b]$ *for all $a, b \in S$.*

Proof. It follows from Corollary 2 that (1) $\sim$ (3) are equivalent, and that (4) $\sim$ (6) are equivalent. It is clear that (1) implies (4). We assume that (4) holds. Let A be any right ideal of S and b any element of S. Let $a=sb$ ($a \in A, s \in S$) be any element of $A \cap Sb$. Then we have

$$a=sb \in [a]_r \cap Sb=[a]_r b \subseteq Ab$$

and so we have

$$A \cap Sb \subseteq Ab$$

for all $b \in S$. This means that A is R-pure in S. Therefore we obtain that (4) implies (1).

3. A semigroup S is called *regular* if, for any element $a \in S$, there exists an element x in S such that $a=axa$.

The equivalence of (1) and (2) in the next theorem is due to the author ([3] Theorem 12). The rest of the proof can be easily proved.

Theorem 4. *For an S-subsystem N of a right S-system M_S over a regular semigroup S the following conditions are equivalent*:

(1) *N is R-pure in S.*

(2) $N \cap Me=Ne$ *for all idempotents $e \in S$.*

(3) $N \cap M[e]=N[e]$ *for all idempotents $e \in S$.*

Corollary 5. *For a right ideal A of a regular semigroup S the following conditions are equivalent*:

(1) *A is R-pure in S.*

(2) $A \cap Se=Ae$ *for all idempotents $e \in S$.*

(3) $A \cap S[e]=A[e]$ *for all idempotents $e \in S$.*

4. A right S-system M_S is called *unital* if S contains an identity 1 such that $x1=x$ for all $x \in M$. A right S-system M_S is called *torsion free* if $xd=yd$ with d cancellable in S implies $x=y$, and is called *divisible* if $Md=M$ for every cancellable element $d \in S$.

Theorem 6. *Let M_S be a divisible torsion free right S-system over a semigroup S. Then any R-pure S-subsystem N of M_S is divisible.*

Proof. Let x be any element of N. Then, since M_S is divisible, there exists an element y in M such that $x=yd$ for every cancellable element $d \in S$. Since N is R-pure in S,

$$x=yd \in N \cap Md=Nd.$$

This implies that there exists an element z in N such that

$$yd=zd.$$

Since M_S is torsion free, we have

$$y=z \in N,$$

and so we have

$$N \subseteq Nd.$$

Since the converse inclusion always holds, we have

$$N=Nd$$

for every cancellable element $d \in S$. Therefore N is divisible.

Theorem 7. *Any S-subsystem N of a unital right S-system M_S over a group S is R-pure and divisible.*

Proof. For any element $a \in S$, we have

$$N = N1 = N(a^{-1}a) = (Na^{-1})a \subseteq Na \subseteq N,$$

and so we have

$$N = Na$$

for all $a \in S$. This holds for all cancellable elements $a \in S$. Thus N is divisible. On the other hand, we have

$$N \cap Ma = Na \cap Ma = Na$$

for all $a \in S$. Therefore N is R-pure in S. This completes the proof of the theorem.

5. A semigroup S is called *normal* if $aS = Sa$ for all $a \in S$ ([4]). Then we have the following.

Theorem 8. *Let M_S be a right S-system over a normal semigroup S. Then the minimal S-subsystem N of M_S is R-pure and divisible.*

Proof. For any element a of S, it follows that

$$Na \subseteq N.$$

Since S is normal, we have

$$(Na)S = N(aS) = N(Sa) = (NS)a \subseteq Na.$$

This means that Na is an S-subsystem of M_S. Then it follows from this and the minimality of N that

$$Na = N$$

for all $a \in S$. Then N is R-pure in S and divisible (see the proof of Theorem 7). This completes the proof of the theorem.

Let A be any right ideal of a normal semigroup S. Then, as is easily seen,

$$AS = SA$$

holds. Thus we have the following lemma.

Lemma 9. *Any one-sided ideal of a normal semigroup is two-sided.*

The following corollary is immediate from Theorem 8 and Lemma 9.

Corollary 10. *The minimal right (left, two-sided) ideal of a normal semigroup is a group.*

6. A semigroup S is called *R-pure-free* if it does not properly contain any R-pure right ideal. In this section we give a non-trivial class of R-pure-free semigroups.

A commutative semigroup S is called *archimedean* if, for any elements a and b of S, there exist elements x and y in S and positive integers m and n such that

$$a^m = xb \quad \text{and} \quad b^n = ya.$$

By an *N-semigroup* we mean a commutative cancellative archimedean semigroup without idempotents. Then we have the following.

Theorem 11. *Any N-semigroup is R-pure-free.*

Proof. Let A be any R-pure right ideal of an N-semigroup S, and let a and s be respectively any elements of A and S. Since S is archimedean, there exist an element x in S and a positive integer m such that

$$a^m = xs.$$

Since A is R-pure in S, we have

$$a^m = sx \in A \cap Sx = Ax.$$

This implies that there exists an element b in A such that

$$sx = bx.$$

Since S is cancellative, we have

$$s = b \in A$$

and so we have

$$S \subseteq A.$$

Therefore we obtain that

$$S = A$$

and that S is R-pure-free. This completes the proof of the theorem.

References

[1] A. H. Clifford and G. B. Preston: The Algebraic Theory of Semigroups, Vol. I. Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I. (1961).

[2] N. Kuroki: Note on purity of ideals on semigroups. Comment. Math. Univ. St. Pauli, **22**, 13–17 (1973).

[3] ——: A characterization of regular monoids (to appear).

[4] Š. Schwarz: A theorem on normal semigroups. Czechoslovak. Math. J., **10**(85), 197–200 (1960).