191. A Note on the Normal Subgroups of 4-fold Transitive Permutation Groups of Degree 5m+4

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1. Introduction. The following is the well-known theorem of Jordan ([8], p. 65).

Theorem (Jordan). *Let G be a k-transitive permutation group on a set Ω, $k\geqq 2$, and G be not the symmetric group on Ω. If $H\neq 1$ is a normal subgroup of G, then H is $(k-1)$-transitive on Ω with the exception when $|\Omega|$ is a power of 2 and $k=3$, and in this exceptional case H may be an elementary abelian 2-group transitive and regular on Ω.*

This theorem is refined by A. Wagner [7], N. Ito [4], M. Aschbacher [1], J. Saxl [6], and E. Bannai [2]. In this note we shall prove the following:

Theorem.[1] *Let G be a 4-fold transitive permutation group on a set Ω, where $|\Omega|=5m+4$ (m: integer) and greater than 5. If $H\neq 1$ is a normal subgroup of G, then H is also 4-fold transitive on Ω.*

2. Proof of the theorem. In order to prove the theorem we need the following:

Lemma. *Let G be a t-fold transitive permutation group on a set Ω for $t\geqq 4$ and let $H\neq 1$ be a normal subgroup of G. Then for all $\Delta\subseteq\Omega$ with $|\Delta|=t$, $H^{\Delta}_{(\Delta)}=S_t$.*

Proof. We omit the proof of the lemma. (See [3].)

Now we start with the proof of the theorem. Suppose $|\Omega|=5m+4$ be the minimal degree >5 such that there is a counterexample to the theorem. Let G be a 4-fold transitive group on Ω of degree $5m+4$ containing a non-trivial normal subgroup H which is 3-fold transitive but not 4-fold transitive on Ω. Let $\alpha,\beta,\gamma\in\Omega$ and let $\Gamma_1,\cdots,\Gamma_k$ be the $H_{\alpha\beta\gamma}$-orbits on Ω-$\{\alpha,\beta,\gamma\}$. Then by assumption $k\geqq 2$, $|\Gamma_1|=\cdots=|\Gamma_k|$ and $|\Gamma_1|$ is not divisible by 5. At first we shall show that there exists a non-trivial 5-element in H fixing at least 4 points of Ω. By the lemma above and a result of Wagner ([7], Lemma 4), $|\Gamma_1|$ must be even. Let $\delta\in\Gamma_1$ and let T be a Sylow 2-subgroup of $H_{\alpha\beta\gamma\delta}$. Then $T\neq 1$ by Theorem 2 of J. King [5]. Now if $T^g\subseteq G_{\alpha\beta\gamma\delta}$ for some $g\in G$ then $T^g\subseteqq G_{\alpha\beta\gamma\delta}\cap H=H_{\alpha\beta\gamma\delta}$ and hence there is an element h of $H_{\alpha\beta\gamma\delta}$ such that $T^g=T^h$. Thus by a well-known lemma of Witt $N_G(T)^{fix\,T}$ is 4-fold transitive.

1) I was informed that the same result was obtained also by E. Bannai (University of Tokyo) independently.

Since $|\Gamma_1|=|H_{\alpha\beta\gamma}:H_{\alpha\beta\gamma\delta}|$ is even, $N_H(T)^{fix\,T}\neq 1$. Then $N_H(T)^{fix\,T}$ is 3-fold transitive by the theorem of Jordan, and using again the above mentioned theorem of King we see that $|fix\ T|$ is 5, 7 or 11 and $N_H(T)^{fix\,T}$ is S_5, A_7 or M_{11}. So $5||N_H(T)^{fix\,T}|$. Hence there is an element $y\in H$ which is of order 5. Since $|\Omega|=5m+4$, y fixes at least 4 points of Ω and hence 5 divides $|H_{\alpha\beta\gamma\delta}|$. Let Q be a Sylow 5-subgroup of $H_{\alpha\beta\gamma\delta}$. Then by the same reason as for T the above $N_G(Q)^{fix\,Q}$ is 4-fold transitive. Since $|\Gamma_1|$ is not divisible by 5, Q is a Sylow 5-subgroup of $H_{\alpha\beta\gamma}$ and hence by a lemma of Witt $N_H(Q)^{fix\,Q}$ is 3-fold transitive, in particular $N_H(Q)^{fix\,Q}\neq 1$. Clearly $|fix\ Q|=5a+4$ (a: integer) and $5a+4<5m+4$. Since $N_H(Q)^{fix\,Q}\lhd N_G(Q)^{fix\,Q}$, $N_H(Q)^{fix\,Q}$ is either 4-fold transitive on $fix\ Q$ or 3-fold transitive on $fix\ Q$ with $|fix\ Q|=4$. In either case we get the following: $fix\ Q-\{\alpha,\beta,\gamma\}\subseteqq\Gamma_1$. Thus Q has no fixed point on Γ_2. Hence 5 divides $|\Gamma_2|=|\Gamma_1|$, which is a contradiction. Thus we complete the proof of the theorem.

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