188. *Singularities of the Riemann Functions of Hyperbolic Mixed Problems in a Quarter-Space*

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Introduction. Matsumura [4] studied singularities of Riemann functions of hyperbolic mixed problems in a quater-space and determined the location of reflected waves by means of "localization theorem". In general Riemann functions also have singularities corresponding to lateral waves and boundary waves (see, Duff [3], Deakin [2]). Lateral waves arise from the presence of branch points in reflection coefficients and boundary waves are caused by real zeros of Lopatinski determinant. In this note we give a localization theorem which determines explicitly the location of lateral waves. The localization theorem of the fundamental solutions for the hyperbolic operators with constant coefficients in the whole space was established by Atiyah, Bott and Gårding [1].

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1. Assumptions and Riemann functions. Let $\boldsymbol{R}^n$ denote the n-dimensional Euclidean space and Ξ^n its complex dual space and write $x'=(x_1, \cdots, x_{n-1})$, $x''=(x_2, \cdots, x_n)$ for the coordinate $x=(x_1, \cdots, x_n)$ in $\boldsymbol{R}^n$ and $\xi'=(\xi_1, \cdots, \xi_{n-1})$, $\xi''=(\xi_2, \cdots, \xi_n)$ for the dual coordinate $\xi=(\xi_1, \cdots, \xi_n)$. The variable x_1 will play the role of "time", the variables $x_2, \cdots, x_n$ will play the role of "space". We shall also denote by $\boldsymbol{R}^n_+$ the half-space $\{x=(x', x_n) \in \boldsymbol{R}^n ; x_n>0\}$. For differentiation we will use the symbol $D=i^{-1}(\partial/\partial x_1, \cdots, \partial/\partial x_n)$.

Let $P=P(\xi)$ be a hyperbolic polynomial of order m of n variables ξ with respect to $\vartheta=(1, 0, \cdots, 0) \in \mathrm{Re}\, \Xi^n$ in the sense of Gårding. We consider the mixed initial-boundary value problem for the hyperbolic operator $P(D)$ in a quater-space

(1) $\quad P(D)u(x)=f(x), \qquad x \in \boldsymbol{R}^n_+,\ x_1>0,$

(2) $\quad (D_1^k u)(0, x'')=0, \qquad 0 \le k \le m-1,\ x_n>0,$

(3) $\quad B_j(D)u(x)|_{x_n=0}=0, \qquad 1 \le j \le l,\ x_1>0.$

Here the $B_j(D)$ are boundary operators with order m_j. The number l of boundary conditions will be determined later on. We assume that the hyperplane $x_n=0$ is non-characteristic for $P(D)$ and $B_j(D)$.

Let $\mathrm{Re}\, A$ be the real hypersurface $\{\xi \in \mathrm{Re}\, \Xi^n ; P^0(\xi)=0\}$, where $P^0(\xi)$

denotes the principal part of $P(\xi)$. Further we denote by $\Gamma=\Gamma(A,\vartheta)$ ($\subset \operatorname{Re}\Xi^n$) the component of $\operatorname{Re}\Xi^n\backslash\operatorname{Re}A$ which contains ϑ. When $\xi'\in\operatorname{Re}\Xi^{n-1}-is\vartheta'-i\Gamma_0$ with s large enough, we can denote the roots of $P(\xi',\lambda)=0$ with respect to λ by $\lambda_1(\xi'),\cdots,\lambda_m(\xi')$, which are enumerated so that

$$(4)\qquad \begin{aligned}&\operatorname{Im}\lambda_k(\xi')>0, && 1\le k\le l,\\ &\operatorname{Im}\lambda_k(\xi')<0, && l+1\le k\le m.\end{aligned}$$

Here Γ_0 denotes the set $\{\eta'\in\operatorname{Re}\Xi^{n-1};\,(\eta',0)\in\Gamma\}$. This number l determines the number of boundary conditions (see [4]). Let $\mu_1(\xi'),\cdots,\mu_m(\xi')$ be the roots of $P^0(\xi',\mu)=0$. Since

$$(5)\qquad t^{-m}P(t\xi',t\mu)\longrightarrow P^0(\xi',\mu)\qquad \text{as } t\to\infty,$$

it follows that, with suitable labelling,

$$(6)\qquad t^{-1}\lambda_k(t\xi')\longrightarrow\mu_k(\xi'),\ 1\le k\le m,\qquad \text{as } t\to\infty.$$

We now define Lopatinski determinant for the system $\{P,B_j\}$ by

$$(7)\qquad R(\xi')=\det\,(B_j(\xi',\lambda_k(\xi'))/\textstyle\prod_{1\le k<i\le l}(\lambda_i(\xi')-\lambda_k(\xi'))$$

and for the system $\{P^0,B_j^0\}$ by

$$(8)\qquad R^0(\xi')=\det\,(B_j^0(\xi',\mu_k(\xi')))/\textstyle\prod_{1\le k<i\le l}(\mu_i(\xi')-\mu_k(\xi')).$$

Here $B_j^0(\xi)$ denotes the principal part of $B_j(\xi)$. We state the assumptions that we impose on $\{P,B_j\}$:

(A. 1) $$P(\xi)=p_1(\xi)\cdots p_q(\xi),$$

where the $p_j(\xi)$ are distinct strictly hyperbolic polynomials with respect to ϑ and irreducible over the complex number field $\boldsymbol{C}$.

(A. 2) For each $p_j^0(\xi)$ and non-zero $\xi'\in\operatorname{Re}\Xi^{n-1}$ the real roots of $p_j^0(\xi',\mu)=0$ are at most double.

(A. 3) If $p_j^0(\xi',\mu)=0$ has real double roots for fixed $\xi'(\neq 0)\in\operatorname{Re}\Xi^{n-1}$, the number of its real double roots is 1 and $p_i^0(\xi',\mu)=0$ has no real double roots for $i\neq j$.

(A. 4) $R(\xi')\neq 0$ when $\xi'\in\operatorname{Re}\Xi^{n-1}-is\vartheta'-i\Gamma_0$ with s large enough. Here $p_j^0(\xi)$ denotes the principal part of $p_j(\xi)$.

Now we can construct the Riemann function $G(x,y)$ for $\{P,B_j\}$ (see [4]). Write

$$(9)\qquad \begin{aligned}&G(x,y)=E(x-y)-F(x,y),\\ &\qquad x\in\boldsymbol{R}^n_+,\ x_1>0,\ y=(0,y_2,\cdots,y_n)\in\boldsymbol{R}^n_+,\end{aligned}$$

where $E(x)$ is the fundamental solution defined by

$$(10)\qquad \begin{aligned}E(x)=(2\pi)^{-n}\int_{\operatorname{Re}\Xi^n}\exp\,[ix\cdot(\xi+i\eta)]P(\xi+i\eta)^{-1}d\xi,\\ \eta\in -s\vartheta-\Gamma.\end{aligned}$$

Then the reflected Riemann function $F(x,y)$ is written in the form

$$(11)\qquad \begin{aligned}F(x,y)=(2\pi)^{-n}\int_{\operatorname{Re}\Xi^{n-1}}&\textstyle\sum_{j=1}^{l}R_j(x_n,\xi'+i\eta')R(\xi'+i\eta')^{-1}\\ &\times\exp\,[i(x'-y')\cdot(\xi'+i\eta')]\\ &\times\left\{\int_{-\infty}^{\infty}\exp\,[-iy_n(\xi_n+i\eta_n)]B_j(\xi+i\eta)P(\xi+i\eta)^{-1}d\xi_n\right\}d\xi'\end{aligned}$$

in the distribution sense with respect to $(x, y) \in \overline{\boldsymbol{R}^n_+} \times \boldsymbol{R}^n_+$. Here the $R_j(x_n, \xi')$ are defined by replacing in $R(\xi')$ the j-th row vector of the determinant with the vector $(\exp[i\lambda_1(\xi')x_n], \cdots, \exp[i\lambda_l(\xi')x_n])$.

2. Localization theorem. According to [1], we introduce the notion of localization of polynomials.

Definition. Let $P(\xi)$ be a polynomial of degree $m \geq 0$ and develop $t^m P(t^{-1}\xi + \zeta)$ in ascending power of t

$$t^m P(t^{-1}\xi + \zeta) = t^p P_\xi(\zeta) + O(t^{p+1}), \tag{12}$$

where $P_\xi(\zeta)$ is the first coefficient that does not vanish identically in ζ. The number $p = m_\xi(P)$ is called the multiplicity of ξ relative to P and the polynomial $\zeta \to P_\xi(\zeta)$ the localization of P at ξ.

Let $\mathrm{D}(P_+)(\xi')$ and $\mathrm{D}(P^0_+)(\xi')$ denote the discriminants of $P_+(\xi', \lambda) = P^0(0,1)\prod_{k=1}^{l}(\lambda - \lambda_k(\xi')) = 0$ in λ and $P^0_+(\xi', \mu) = P^0(0,1)\prod_{k=1}^{l}(\mu - \mu_k(\xi'))=0$ in μ, respectively. We assume that $\xi_0 = (\xi_{01}, \cdots, \xi_{0n}) \in \mathrm{Re}\,\Xi^n$ satisfies the following conditions: (i) $\xi_0 \in \mathrm{Re}\,A$. (ii) $\mathrm{D}(P^0_+)(\xi'_0) \neq 0$. (iii) There exists $b, 1 \leq b \leq l$, such that $\mu_b(\xi'_0)$ is a real double root of $p^0_1(\xi'_0, \mu) = 0$. (iv) $\xi_{0n} \neq \mu_b(\xi'_0)$. (v) $R^0(\xi'_0) \neq 0$. Moreover we choose a number k such that (vi) $\mu_k(\xi'_0)$ is real and $k \neq b$, $1 \leq k \leq l$. Then there exists a unique number r such that

$$p^0_r(\xi', \mu_k(\xi')) = 0 \qquad \text{for } \xi' = \xi'_0 - it^{-1}\vartheta. \tag{13}$$

Put with δ small positive

$$\begin{aligned} \alpha_k(\xi'_0) = (2\pi i)^{-1} \int_{|z-\mu_k(\xi')|=\delta} & \\ \times z\{p^0_r(\xi'_0, z)\partial p^1_r/\partial z(\xi'_0, z) &- \partial p^0_r/\partial z(\xi'_0, z)\cdot p^1_r(\xi'_0, z)\} \\ \times \{p_r(\xi'_0, z)\}^{-2} dz, & \end{aligned} \tag{14}$$

where $p^1_r(\xi', z)$ denotes the principal part of $p_r(\xi', z) - p^0_r(\xi', z)$. Moreover put

$$\begin{aligned} F_{\xi_0,k}(x, y) &= (2\pi)^{-n}\textstyle\sum_{j=1}^{l} \varDelta_{jk}(\xi'_0)B^0_j(\xi_0)\cdot\varDelta(\xi'_0)^{-1}\exp[i\alpha_k(\xi'_0)x_n] \\ &\times \int_{\mathrm{Re}\,\Xi^{n-1}} \exp[i(x' - y' + x_n\,\mathrm{grad}_{\xi'}\,\mu_k(\xi'_0))\cdot(\zeta' + i\eta')] \\ &\times \left\{\int_{-\infty}^{\infty} \exp[-iy_n(\zeta_n + i\eta_n)]P_{\xi_0}(\zeta + i\eta)^{-1} d\zeta_n\right\} d\zeta', \end{aligned} \tag{15}$$

$$\begin{aligned} F_{\xi_0,kb}(x, y) &= (2\pi)^{-n}\textstyle\sum_{j=1}^{l}\{\varDelta'_{jk}(\xi'_0)\varDelta(\xi'_0)^{-1} + \varDelta_{jk}(\xi'_0)\varDelta'(\xi'_0)\cdot\varDelta(\xi'_0)^{-2}\}B^0_j(\xi_0) \\ &\times \exp[i\alpha_k(\xi'_0)x_n]\int_{\mathrm{Re}\,\Xi^{n-1}} \sqrt{\mathrm{grad}_{\xi'}\,\rho^0(\xi'_0)\cdot(\zeta' + i\eta') + \beta(\xi'_0)} \\ &\times \exp[i(x' - y' + x_n\,\mathrm{grad}_{\xi'}\,\mu_k(\xi'_0))\cdot(\zeta' + i\eta')] \\ &\times \left\{\int_{-\infty}^{\infty} \exp[-iy_n(\zeta_n + i\eta_n)]P_{\xi_0}(\zeta + i\eta)^{-1} d\zeta_n\right\} d\zeta'. \end{aligned} \tag{16}$$

Here $\varDelta(\xi'_0)$ denotes the determinant $(B^0_q(\xi'_0, \mu_r(\xi'_0)))$ and $\varDelta_{jk}(\xi'_0)$ its (j, k)-cofactor, and $\varDelta'(\xi'_0)$ is defined by replacing in $\varDelta(\xi'_0)$ the b-th column vector with ${}^t(\partial B^0_1/\partial\xi_n(\xi'_0, \mu_0(\xi'_0)), \cdots, \partial B^0_l/\partial\xi_n(\xi'_0, \mu_0(\xi'_0)))$ and $\varDelta'_{jk}(\xi'_0)$ denotes its (j, k)-cofactor. $\rho^0(\xi'_0)$ and $\beta(\xi'_0)$ will be given by (25) and (34) in Section 3, respectively. Then we have following

Localization theorem. *Assume that the conditions* (A. 1)–(A. 4) *are satisfied and that* ξ_0 *and* k *satisfy the above conditions* (i)–(vi). *Put* $p=m_{\xi_0}(P)$. *Then*

$$t^{m-p+1/2}\exp\left[-it\{(x'-y')\cdot\xi_0'+x_n\mu_k(\xi_0')-y_n\xi_{0n}\}\right]F(x,y) \\ -t^{1/2}F_{\xi_0,k}(x,y)\longrightarrow F_{\xi_0,kb}(x,y) \qquad \text{as } t\to\infty \tag{17}$$

in the distribution sense with respect to $(x,y)\in \boldsymbol{R}^n_+\times\boldsymbol{R}^n_+$ *and*

$$\operatorname{supp}_{(x,y)} F_{\xi_0,kb}(x,y)\subset \operatorname{sign}\operatorname{supp}_{(x,y)} F(x,y). \tag{18}$$

Moreover

$$\operatorname{supp}_{(x,y)} F_{\xi_0,kb}\subset\{(x,y)\in \boldsymbol{R}^n_+\times\boldsymbol{R}^n_+\,;\, y=(0,y_2,\cdots,y_n), \\ [x'-y'+x_n \operatorname{grad}_{\xi'}\mu_k(\xi_0')]\cdot\eta'-y_n\eta_n\geq 0,\ \eta\in\Gamma_{\xi_0,b}\}, \tag{19}$$

where $\Gamma_{\xi_0,b}=\Gamma(A_{\xi_0,b},\vartheta)$ *and* $\operatorname{Re}A_{\xi_0,b}=\{\zeta\in\operatorname{Re}\Xi^n\,;\,P^0_{\xi_0}(\zeta)\cdot[\operatorname{grad}_{\xi'}\rho^0(\xi_0')\cdot\zeta']=0\}$.

Remark 1. Matsumura [4] showed that

$$\operatorname{supp}_{(x,y)} F_{\xi_0,k}\subset \operatorname{sing}\operatorname{supp}_{(x,y)} F. \tag{20}$$

This result determines the location of reflected waves. Our result determines the location of lateral waves.

Remark 2. The condition (A. 3) can be removable.

3. Outline of proof. If t is chosen sufficiently large, then we can write $\lambda_b(t\xi_0'+\zeta'+i\eta')$ in the form

$$\lambda_b(t\xi_0'+\zeta'+i\eta')=\sigma(t\xi_0'+\zeta'+i\eta')+\sqrt{\rho(t\xi_0'+\zeta'+i\eta')} \\ \text{for fixed } \zeta+i\eta\in\operatorname{Re}\Xi^n-is\vartheta-i\Gamma, \tag{21}$$

where $\operatorname{Im}\sqrt{\rho(t\xi_0'+\zeta'+i\eta')}>0$ and $\sigma(\cdot)$ and $\rho(\cdot)$ are analytic for $|\zeta'+i\eta'|\leq c_1 t$. In fact, $\lambda_b(t\xi_0'+\zeta'+i\eta')$ is a root of the equation

$$\lambda^2-b_1(\cdot)\lambda+2^{-1}(b_1(\cdot)^2-b_2(\cdot))=0, \tag{22}$$

where

$$b_1(t\xi_0'+\zeta'+i\eta')=(2\pi i)^{-1}\int_{|z-\mu_b(\xi_0')|=\delta} t^2z\cdot\partial p_1/\partial z(\cdot,tz)\cdot p_1(\cdot,tz)^{-1}dz, \tag{23}$$

$$b_2(t\xi_0'+\zeta'+i\eta')=(2\pi i)^{-1}\int_{|z-\mu_0(\xi_0')|=\delta} t^3z^2\cdot\partial p_1/\partial z(\cdot,tz)\cdot p_1(\cdot,tz)^{-1}dz. \tag{24}$$

Similarly we have

$$\mu_0(t\xi_0'+\zeta'+i\eta')=\sigma^0(t\xi_0'+\zeta'+i\eta')+\sqrt{\rho^0(t\xi_0'+\zeta'+i\eta')}. \tag{25}$$

Put

$$F_{\xi_0,kb}(x,y\,;t)=t^{m-p+1/2}\exp\left[-it\{(x'-y')\cdot\xi_0'+x_n\mu_k(\xi_0')-y_n\xi_{0n}\}\right] \\ \times F(x,y)-t^{1/2}F_{\xi_0,k}(x,y). \tag{26}$$

The integrals over $|\zeta+i\eta|\geq c_2t^{1/N}$ on the right hand side of (26) tend to zero in the distribution sense as $t\to\infty$, where N is chosen large enough and η is fixed so that $\eta\in -s\vartheta-\Gamma$ and $(\eta',0)\in -s\vartheta-\Gamma$. The term corresponding to the integrals over $|\zeta+i\eta|\leq c_2t^{1/N}$ in (26) is written in the form

$$\tilde F_{\xi_0,kb}(x,y\,;t) \\ =(2\pi)^{-n}\int t^{1/2}\textstyle\sum_{j,h=1}^{l}\operatorname{cof}_{jh}(B_q(t\xi_0'+\zeta'+i\eta',\lambda_r(t\xi_0'+\zeta'+i\eta'))) \\ \times\det(B_q(\cdot,\lambda_r(\cdot)))^{-1}\exp[i(x'-y')\cdot(\zeta'+i\eta')]$$

$$\times \exp [ix_n t\{\lambda_h(t\xi_0'+\zeta'+i\eta')/t-\mu_k(\xi_0')\}]$$

$$\times\left\{\int_{-\infty}^{\infty} \exp[-iy_n(\zeta_n+i\eta_n)]B_j(t\xi_0+\zeta+i\eta)\right. \tag{27}$$

$$\left.\times\left\{t^{p-m}P(t\xi_0+\zeta+i\eta)\right\}^{-1}d\zeta_n\right\}d\zeta'-(2\pi)^{-n}t^{1/2}\varDelta_{jk}(\xi_0')B_j^0(\xi_0)\varDelta(\xi_0')^{-1}$$

$$\times \exp[i\alpha_k(\xi_0')x_n]\int \exp[i(x'-y'+x_n \operatorname{grad}_{\xi'}\mu_k(\xi_0'))\cdot(\zeta'+i\eta')]$$

$$\times\left\{\int_{-\infty}^{\infty} \exp[-iy_n(\zeta_n+i\eta_n)]P_{\xi_0}(\zeta+i\eta)^{-1}d\zeta_n\right\}d\zeta'.$$

When $|\zeta+i\eta|\le c_2t^{1/N}$, we have

$$\operatorname{cof}_{jk}(B_q(\cdot,\lambda_r(\cdot)))\cdot\det(B_q(\cdot,\lambda_r(\cdot)))^{-1}=\varLambda_{1j}(\cdot)+\varLambda_{2j}(\cdot)\sqrt{\rho(\cdot)}, \tag{28}$$

where $\varLambda_{1j}(\cdot)$ and $\varLambda_{2j}(\cdot)$ are analytic for $|\zeta'+i\eta'|\le c_2t^{1/N}$. If the condition (A. 3) is removed, (28) does not hold in general. However, by obvious modifications we can prove our theorem. For $|\zeta+i\eta|\le c_2t^{1/N}$ we have

$$t^{-1}\rho(t\xi_0'+\zeta'+i\eta')=\operatorname{grad}_{\xi'}\rho^0(\xi_0')\cdot(\zeta'+i\eta')+\beta(\xi_0')+0(t^{-1+2/N}), \tag{29}$$

$$\begin{aligned} &t^{-m_j}B_j(t\xi_0+\zeta+i\eta)/\{t^{p-m}P(t\xi_0+\zeta+i\eta)\}\\ &\quad=B_j^0(\xi_0')\cdot P_{\xi_0}(\zeta+i\eta)^{-1}+0(t^{-1+1/N}),\end{aligned} \tag{30}$$

$$t^{m_j}\varLambda_{1j}(\cdot)=\varDelta_{jk}(\xi_0')\varDelta(\xi_0')^{-1}+0(t^{-1+1/N}), \tag{31}$$

$$t^{m_j+1}\varLambda_{2j}(\cdot)=\varDelta'_{jk}(\xi_0')\varDelta(\xi_0')^{-1}+\varDelta_{jk}(\xi_0')\varDelta'(\xi_0')\varDelta(\xi_0')^{-2}+0(t^{-1+1/N}), \tag{32}$$

$$\begin{aligned} &ix_nt(\lambda_k(t\xi_0'+\zeta'+i\eta')/t-\mu_k(\xi_0'))\\ &\quad=ix_n(\operatorname{grad}_{\xi'}\mu_k(\xi_0')\cdot(\zeta'+i\eta')+\alpha_k(\xi_0'))+0(t^{-1+2/N}),\end{aligned} \tag{33}$$

where

$$\begin{aligned}\beta(\xi_0')=(2\pi i)^{-1}\int_{|z-\mu_b(\xi_0')|=\delta} &1/2\cdot z(z-2\mu_b(\xi_0'))\\ &\times\{\partial p_1^1/\partial z(\xi_0',z)\cdot p_1^0(\xi_0',z)-p_1^1(\xi_0',z)\cdot\partial p_1^0/\partial z(\xi_0',z)\}\{p_1^0(\xi_0',z)\}^{-2}dz.\end{aligned}$$

From (29)–(33) it follows that

$$\tilde F_{\xi_0,kb}(x,y,t)\longrightarrow F_{\xi_0,kb}(x,y)\qquad \text{as } t\to\infty. \tag{35}$$

Moreover it follows from the conditions (A. 1) and (A. 2) that $\operatorname{grad}_{\xi'}\rho^0(\xi_0')$ is a real vector and that

$$\operatorname{grad}_{\xi'}\rho^0(\xi_0')\cdot\vartheta'\neq 0, \tag{36}$$

i.e. $\zeta\to\operatorname{grad}_{\xi'}\rho^0(\xi_0')\cdot\zeta'$ is a hyperbolic polynomial with respect to ϑ. This completes the proof.

The detailed proof and some examples will be given in a forthcoming paper.

References

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