183. *On Certain Complex Structures on the Product of Two Odd Dimensional Spheres*

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§ 1. The purpose of this note is to report our recent results on the complex structures on the product of two spheres of odd dimensions. The details will appear elsewhere.

Calabi and Eckmann [1] constructed a complex structure on $S^{2p-1} \times S^{2q-1}$ $(p,q\geqq 2)$. We will show that, under certain additional conditions, a complex manifold homeomorphic to $S^{2p-1}\times S^{2q-1}$ is a finite abelian branched covering of a submanifold in the above-mentioned Calabi-Eckmann manifold, where p and q are greater than one.

Throughout this note, $\mathcal{O}$ (or $\mathcal{O}_X$) denotes the structure sheaf of a complex manifold X, $q=q(X)=\dim_C H^1(X,\mathcal{O})$ its irregularity, and $b_\nu=b_\nu(X)$, the ν-th Betti number.

§ 2. The construction of Calabi-Eckmann manifolds. We construct a Calabi-Eckmann manifold as follows: For each $t\in C$, let g_t^λ be the biholomorphic automorphism of $(C^p-(o))\times(C^q-(o))$ which maps (z,w) to $(z\exp(t), w\exp(\lambda t))$, where λ is a fixed complex number with $\operatorname{Im}\lambda\neq 0$. Let G be the one-parameter complex Lie group consisting of g_t^λ's. G operates freely and properly on $(C^p-(o))\times(C^q-(o))$. Hence by Holmann [2], we can construct the quotient manifold $M=(C^p-(o))\times(C^q-(o))/G$, which is a compact complex manifold of dim. $(p+q-1)$. M is called a Calabi-Eckmann manifold. Note that M is a complex analytic fibre bundle over $P^{p-1}\times P^{q-1}$ of which the fibre is an elliptic curve. Moreover M has a holomorphic torus action.

§ 3. A lemma on the structure of elliptic *n*-folds. We need the following lemma.

Lemma. *Let V^n and $\tilde{D}^{n-1}$ be an n-dimensional complex manifold and an $(n-1)$-dimensional polydisk, respectively. Let $\pi: V^n\rightarrow\tilde{D}^{n-1}$ be a flat proper holomorphic mapping, whose general fibres are biholomorphically equivalent to a fixed elliptic curve E. Suppose that there exists a subvariety S of $\tilde{D}$ with codimension greater than 1, such that, outside of S, every fibre of π is a non-singular elliptic curve. Then there exists a polydisk $D\subset\tilde{D}$ such that every fibre of $\pi|_D$ is a non-singular elliptic curve having the general fibre E as a finite unramified covering, and that the singular locus, i.e., the image of the critical*

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points of $\pi|_D$, *has only simple normal crossings as its singularities. Moreover* $\pi^{-1}(D)$ *has a holomorphic semi-free action of complex* 1-*dimensional torus* T, *such that the natural projection* $\pi^{-1}(D)\rightarrow\pi^{-1}(D)/T$ *coincides with* π.

Outline of proof. First we see that there exists a subvariety W of $\tilde{D}$ of codimension 1, such that $V-\pi^{-1}(W)\rightarrow\tilde{D}-W$ is a complex analytic fibre bundle with E as fibre. Since, by assumption, $\pi^{-1}(p)$ is an elliptic curve at every general point p of W, the monodromy at p around W is trivial. Therefore we can patch up the local torus actions on $V-\pi^{-1}(W)$ and on $\pi^{-1}(W-S)$, and construct a complex torus action on $V-\pi^{-1}(S)$. Since, by the flatness of π, the codimension of $\pi^{-1}(S)$ in V is greater than 1, we can extend this torus action to the whole V by integrating the corresponding holomorphic vector field. This action satisfies the conditions in Holmann [2]. Hence, by shrinking $\tilde{D}$ to an appropriately small subpolydisk D, if necessary, we can find a holomorphic slice Δ in $V|_D$ such that Δ is isomorphic to a polydisk, and π makes Δ naturally a finite abelian branched covering of D, and that $V|_D$ is nothing but the quotient space $D\times E/G$, where G is a finite abelian group acting on $D\times E$ corresponding to the covering transformation group of Δ over D. Then the above lemma follows easily. q.e.d.

Remark 1. The assumption that the general fibre is 1-dimensional is superficial. We can extend the above lemma easily to the case in which every general fibre is an arbitrary dimensional complex torus isomorphic to each other.

Remark 2. Similarly, by using a C^∞-torus action instead of a holomorphic one, we can prove the following.

Proposition. *Let* V^{n+2} *be an* $(n+2)$-*dimensional complex manifold, and let* $\pi: V^{n+2}\rightarrow D$, D *being a bidisk, be a flat proper holomorphic map whose general fibres are* n-*dimensional complex tori which are not necessarily of constant moduli. Suppose that the Jacobian of* π *attains the maximal rank except possibly on* $\pi^{-1}(o)$. *Then* π *is actually smooth.*

§4. Statements of the main results. Let V be an n-dimensional compact complex manifold homeomorphic to $\boldsymbol{S}^{2p-1}\times\boldsymbol{S}^{2q-1}$ $(p+q-1\geqq 3)$, and suppose that there exists a flat holomorphic map π from V onto a certain projective algebraic manifold W. Then the following theorems hold.

Theorem 1. *Suppose that* $p\geqq 2$ *and* $q\geqq 2$. *Then there exists a finite holomorphic map from* V *onto a submanifold* V' *of a Calabi-Eckmann manifold* X, *such that the following diagramme commutes*:

$$\begin{array}{ccccc} V & \longrightarrow & V' & \hookrightarrow & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varphi|_{V'}} & & \downarrow{\scriptstyle\varphi} \\ W & \cong & W & \hookrightarrow & \boldsymbol{P}^{N_1}\times\boldsymbol{P}^{N_2} \end{array}$$

where φ is the natural projection. Moreover W has the same rational homology group as $\boldsymbol{P}^{p-1}\times\boldsymbol{P}^{q-1}$, every fibre of π is a non-singular elliptic curve, the singular locus of π has only simple normal crossings as its singularities, and the moduli of all the general fibres of π are the same. Furthermore, V is an abelian branched covering of V′, and has a complex torus action compatible with that of X.

Theorem 2. *Suppose that $p=1$. Then there exists a finite holomorphic map from V onto a submanifold V′ of a Hopf manifold X, such that the following diagramme commutes:*

$$\begin{array}{ccccc} V & \longrightarrow & V' & \hookrightarrow & X \\ \pi\downarrow & & \downarrow\varphi|_{V'} & & \downarrow\varphi \\ W & \cong & W & \hookrightarrow & \boldsymbol{P}^N \end{array}$$

In this case W has the same rational homology group as $\boldsymbol{P}^{n-1}$, and π has the same properties as in Theorem 1.

Outline of proof of Theorem 1. First we need three lemmata which hold under the assumption of Theorem 1.

Lemma 1. *V and W have the following numerical invariants: $q(V)=1$, $b_1(W)=0$, and $b_2(W)=2$.*

We can prove this using Leray spectral sequences for $\mathcal{O}_V$ and for the constant sheaf $\boldsymbol{C}_V$, corresponding to the fibre space $\pi: V\to W$.

Lemma 2. $H^1(V,\mathcal{O})\cong H^1(V,\mathcal{O}^*)\cong\boldsymbol{C}$.

Proof. Since $V\cong\boldsymbol{S}^{2p-1}\times\boldsymbol{S}^{2q-1}$, it follows that $H^1(V,\boldsymbol{Z})=H^2(V,\boldsymbol{Z})=0$. Therefore, from the exact sequence

$$\cdots\to H^1(V,\boldsymbol{Z})\to H^1(V,\mathcal{O})\to H^1(V,\mathcal{O}^*)\to H^2(V,\boldsymbol{Z})\to\cdots$$

and Lemma 1, the conclusion follows.

Lemma 3. *There exist two very ample line bundle L_1 and L_2 on W such that $c_1(L_1)$ and $c_1(L_2)$ are linearly independent in $H^2(W,\boldsymbol{Z})$, and that $\pi^*(L_1)$ and $\pi^*(L_2)$ generate a lattice in $H^1(V,\mathcal{O}^*)\cong\boldsymbol{C}$, where $c_1(L)$ denotes the first Chern class of a line bundle L.*

Proof is omitted. By these lemmata, we can prove Theorem 1 as follows: First we choose coordinate coverings $\{U_i^\mu\}$ of V and $\{W_i\}$ of W such that $\pi(U_i^\mu)=W_i$. Let $\ell_{\alpha,ij}$ be the transition functions of the line bundle L_α ($\alpha=1,2$) with respect to $\{W_i\}$. Then by Lemmata 2 and 3, we can find non-vanishing holomorphic functions p_i^μ and q_i^μ on each U_i^μ, a holomorphic function $\lambda_{ij}^{\mu\nu}$ on each $U_i^\mu\cap U_j^\nu$ and a complex number c with Im $c\neq 0$ such that the following equations are satisfied:

$$\pi^*\ell_{1,ij}=p_i^\mu\exp(\lambda_{ij}^{\mu\nu})(p_j^\nu)^{-1},$$
$$\pi^*\ell_{2,ij}=q_i^\mu\exp(c\lambda_{ij}^{\mu\nu})(q_j^\nu)^{-1},$$

where we put $\ell_{\alpha,ij}=1$ for $i=j$.

Since L_α is very ample, we can find a basis $\{\zeta_{\alpha,i}^{(k)}\}$ ($k=1,\cdots,N$) of $H^0(W,\mathcal{O}(L_\alpha))$ which gives an imbedding of W into $\boldsymbol{P}^{N_\alpha-1}$, where $N_\alpha=\dim H^0(W,\mathcal{O}(L_\alpha))$. Then by the above equations we can construct a holomorphic map from V into $(\boldsymbol{C}^{N_1}-(o))\times(\boldsymbol{C}^{N_2}-(o))/G$, which maps $x\in U_i^\mu$ to $[(p_i^\mu\pi^*\zeta_{1,i}^{(1)},\cdots,p_i^\mu\pi^*\zeta_{1,i}^{(N_1)},q_i^\mu\pi^*\zeta_{2,i}^{(1)},\cdots,q_i^\mu\pi^*\zeta_{2,i}^{(N_2)})]$ where $G=\{g_t^c\}$ is the one parameter group defined in § 2. The second manifold is nothing but a Calabi-Eckmann manifold and the rest of the theorem can easily be proved from the construction. q.e.d.

The proof of Theorem 2 is similar.

Corollary. *Under the assumptions of Theorem* 1 (*Theorem* 2, *respectively*) V *has the same* $h^{p,q}$ *as the standard Calabi-Eckmann manifold* (*Hopf manifold, respectively*) *of the same dimension.*

References

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[3] M. Kato: Complex structures on $S^1\times S^5$. Proc. Japan Acad., **49**(8), 575–577 (1973).