178. *Lipschitz Functions and Convolution*

By Yoshikazu UNO
Kanazawa University

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1. Introduction. In this paper we shall consider functions defined on the torus. S. Bernstein's theorem [7; vol. 1, p. 240] says that the set Lip α is contained in the space A of functions with an absolutely convergent Fourier series when $\alpha > 1/2$. As is well known, the space A coincides with the space $L^2 * L^2$ [7; vol. 1, p. 251]. These assert that Lip α is contained in $L^2 * L^2$ if $\alpha > 1/2$. On the other hand, R. Salem's result [6] implies that the space $L^1 * L^\infty$ is equal to the space C of all continuous functions (see also [2]). Therefore it is trivial that Lip α is contained in $L^1 * L^\infty$ for $\alpha > 0$. Then it is expected that Lip α is contained in $L^p * L^q$ if $\alpha > 1/q$ where $1 < p < 2$ and $1/p + 1/q = 1$. This fact is proved by using results of N. Aronszajn-K. T. Smith and A. P. Calderon (see [3]). We shall give an elementary proof.

Theorem 1. *Let $1 \leqq p < \infty$, $1/p + 1/q = 1$ and $1 \leqq r \leqq \infty$. If $f \in L^r$ and $\|\sigma_n - f\|_r = O(n^{-\alpha})$ for some $\alpha > 1/q$, then $f \in L^p * L^r$ where σ_n is the n-th $(C, 1)$ mean of Fourier series of f.*

Corollary 1. *Let $1 \leqq p \leqq 2$ and $1/p + 1/q = 1$. If $\alpha > 1/q$, then* Lip α *is contained in $L^p * L^q$. There exists however a function which belongs to* Lip $1/q$ *but not to $L^p * L^q$ if $p \neq 1$.*

Now we denote by BV_p the space of functions of p-bounded variation for $1 \leqq p \leqq \infty$ (see [3] or [5] for definition). It is obvious that BV_1 is the set of functions of ordinary bounded variation and BV_∞ is of bounded functions.

Corollary 2. *If $1 \leqq p \leqq 2$ and $1/p + 1/q = 1$, then the intersection of* Lip α *and $BV_{q-\varepsilon}$ is contained in $L^p * L^q$ for $\alpha > 0$ and $\varepsilon > 0$.*

The case $p = 2$ and $\varepsilon = 1$ is A. Zygmund's theorem [7; vol. 1, p. 241] by $A = L^2 * L^2$ and the case $p = 1$, as previously stated, is trivial from R. Salem's result.

In the proof of Theorem 1, we use a method of R. Salem [6].

2. Lemmas. We shall here state some lemmas.

Lemma 1. *Let $1 \leqq p \leqq \infty$ and $1/p + 1/q = 1$. If a positive and convex sequence $\{\lambda_n\}$ tending to zero satisfies the condition*

$$\sum_{n=1}^{\infty} n^{1+1/q} (\lambda_{n-1} + \lambda_{n+1} - 2\lambda_n) < \infty,$$

then there is a function g in L^p such that $\hat{g}(n) = \lambda_{|n|}$ for every integer n.

Proof. Denoting the Fejér kernel by K_n, the series

$$\sum_{n=1}^{\infty} n(\lambda_{n-1}+\lambda_{n+1}-2\lambda_n)K_n$$

converges in L^p by hypotheses since $\|K_n\|_p=O(n^{1/q})$. Then its sum g is a required function.

Lemma 2. *Let $1\leqq p\leqq\infty$ and $1/p+1/q=1$. If $\beta>1/q$, then there exists a function g in L^p such that $\hat{g}(n)=(|n|+1)^{-\beta}$ for every n.*

Proof. It is a trivial result of Lemma 1.

Lemma 3. *Let $1<p\leqq 2$ and $1/p+1/q=1$. If $f\in L^p*L^2$, then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|\,|n|^{1/q-1/2}<\infty.$$

Proof. Let $f=g*h$ for some g in L^p and h in L^2. Then, by Hölder's inequality, we have

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|\,|n|^{1/q-1/2}$$
$$\leqq\left\{\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^p\,|n|^{p-2}\right\}^{1/2p}\left\{\sum_{n=-\infty}^{\infty} |\hat{g}(n)|^q\right\}^{1/2q}\left\{\sum_{n=-\infty}^{\infty} |\hat{h}(n)|^2\right\}^{1/2}.$$

The three series converge in virtue of theorems of Hardy-Littlewood [7; vol. 2, p. 109], Hausdorff-Young [7; vol. 2, p. 101] and Parseval, respectively.

3. Proof of Theorem 1. We take β such that $1/q<\beta<\alpha$ and a function g as in Lemma 2 for this β. Then $g\in L^p$. Let $\mu_n=1/\hat{g}(n)$ and $\tau_n(x)$ be the n-th $(C,1)$ mean of the series

$$\sum_{n=-\infty}^{\infty} \mu_n\hat{f}(n)\,e^{inx}.$$

Then, summing by parts twice, we have

$$\tau_n=\sum_{k=0}^{n-1}\Delta^2\mu_k(k+1)\sigma_k-(n+1)^{-1}\sum_{k=0}^{n-1}\Delta^2\mu_k(k+1)k\sigma_k$$
$$+2(n+1)^{-1}\sum_{k=0}^{n-1}\Delta\mu_{k+1}(k+1)\sigma_k+\mu_n\sigma_n$$

where $\Delta\mu_k=\mu_k-\mu_{k+1}$ and $\Delta^2\mu_k=\Delta\mu_k-\Delta\mu_{k+1}$. Substituting σ_k-f for σ_k in the above equality, we obtain

$$\tau_n=\sum_{k=0}^{n-1}\Delta^2\mu_k(k+1)(\sigma_k-f)-(n+1)^{-1}\sum_{k=0}^{n-1}\Delta^2\mu_k(k+1)k(\sigma_k-f)$$
$$+2(n+1)^{-1}\sum_{k=0}^{n-1}\Delta\mu_{k+1}(k+1)(\sigma_k-f)+\mu_n(\sigma_n-f)+\mu_0 f$$
$$=\textstyle\sum_1(n)-\sum_2(n)+2\sum_3(n)+\mu_n(\sigma_n-f)+\mu_0 f \qquad \text{say.}$$

By hyposeses, $\Delta\mu_k=O(k^{\beta-1})$ and $\Delta^2\mu_k=O(k^{\beta-2})$, we have

$$\Delta^2\mu_k(k+1)k\,\|\sigma_k-f\|_r=O(k^{\beta-\alpha}),\qquad \Delta\mu_{k+1}(k+1)\,\|\sigma_k-f\|_r=O(k^{\beta-\alpha})$$

and $\mu_n\|\sigma_n-f\|_r=O(n^{\beta-\alpha})$. Hence $\sum_2(n)$ and $\sum_3(n)$ tend to zero in L^r since they are n-th $(C,1)$ means of $\Delta^2\mu_k(k+1)k(\sigma_k-f)$ and $\Delta\mu_{k+1}(k+1)$ (σ_k-f), respectively. Moreover we have

$$\sum_{k=0}^{n-1}\Delta^2\mu_k(k+1)\,\|\sigma_k-f\|_r=O\left(\sum_{k=1}^{n}k^{-1-\alpha+\beta}\right)=O(1)$$

and then $\sum_1(n)$ converges in L^r. Thus τ_n converges in L^r to a function

h. It is easy to see that $\hat{h}(n)=\mu_n\hat{f}(n)$. Therefore $\hat{f}(n)=\hat{g}(n)\hat{h}(n)$, that is, $f=g*h$. Consequently $f\in L^p*L^r$.

4. Proof of Corollary 1. We shall prove the first part. Let $1>\alpha>1/q$ and $f\in\mathrm{Lip}\,\alpha$. It is well known that $\|\sigma_n-f\|_\infty=O(n^{-\alpha})$ [7; vol. 1, p. 91]. Therefore $f\in L^p*L^\infty$ by Theorem 1 and so $f\in L^p*L^q$.

Next we shall show the second part. It is enough to show that there exists a function in $\mathrm{Lip}\,1/q$ but not in L^p*L^2 when $p\neq 1$. We consider the function

$$f(x)=\sum_{n=1}^{\infty}\frac{e^{in\log n}}{n^{1/2+1/q}}e^{inx}.$$

Then $f\in\mathrm{Lip}\,1/q$ and $|\hat{f}(n)|=n^{-1/2-1/q}$ for $n\geqq 1$ [7; vol. 1, p. 197] and so

$$\sum_{n=1}^{\infty}|\hat{f}(n)|\,n^{1/q-1/2}=\infty.$$

Consequently $f\notin L^p*L^2$ by Lemma 3.

Remark. Let $1\leqq p,\ r\leqq 2$. Hölder's and Hausdorff-Young inequalities imply that if $f\in L^p*L^r$, then $\hat{f}\in l^s$ where $1/s=2-1/p-1/r$. Therefore the case $r=2$ is the fact that $\hat{f}\in l^{2p/(3p-2)}$ for every $f\in L^p*L^2$. If f is the function in the proof of the second part of Corollary 1, then $\hat{f}\notin l^{2p/(3p-2)}$ and so this together with the above fact proves the second part of Corollary 1, too. Moreover the part is also proved by using the Rudin-Shapiro polynomials.

5. Proof of Corollary 2. It is trivial when $p=1$ by Corollary 1. Let $1<p\leqq 2$ and $f\in\mathrm{Lip}\,\alpha\cap BV_{q-\varepsilon}$ for some $\alpha,\varepsilon>0$. Then we obtain (see [3] or [5])

$$\int_0^{2\pi}|f(x+y)-f(x)|^{q-\varepsilon}\,dx=O(|y|).$$

Now, by $f\in\mathrm{Lip}\,\alpha$, we have

$$\|f_y-f\|_q=O(|y|^{(1+\alpha\varepsilon)/q}).$$

Therefore $f\in L^p*L^q$ by Theorem 1.

Remark. It is easy to see that BV_q contains $\mathrm{Lip}\,1/q$. Therefore Corollary 2 does not hold when $\alpha=1/q$ and $\varepsilon=0$.

6. Application. The space of all multiplies of type (r,s) will be denoted by M_r^s. L.-S. Hahn [4] showed that if $1\leqq p\leqq q\leqq\infty$ and $1/p+1/q\geqq 1$, then L^p*L^q is contained in M_r^s where

$$1/r=(1-2\theta)+(4\theta-1)/2p+1/2q,$$
$$1/s=2(1-\theta)+(4\theta-3)/2p-1/2q,$$

for all θ, $0\leqq\theta\leqq 1$. He asked, then, whether bounds of θ are improved. If we put $p_2=r_2=1$ and $q_2=s_2=2$ in Hahn's proof of the above result, we can show the following.

Theorem 2. *If $1\leqq p\leqq 2\leqq q\leqq\infty$ and $1/p+1/q\geqq 1$, then L^p*L^q is contained in M_r^s where*

$$1/r=(1/2-\theta)+\theta/p+(1-\theta)/q,$$
$$1/s=(3/2-\theta)-(1-\theta)/p-\theta/q,$$

for all θ, $0 \leqq \theta \leqq 1$.

This Theorem 2 contains the Hahn's result if $q \geqq 2$. Bounds of θ in Theorem 2 cannot be improved. For let σ be s corresponding to $\theta < 0$ in the equality of Theorem 2 and then $3/2 - 1/p < 1/\sigma \leqq 1$. If we take ρ such that $1/\sigma = 3/2 - 1/\rho$, then $p < \rho \leqq 2$. By Remark in § 4, there is a function f in Lip $1/\rho'$ with $\hat{f} \notin l^{\sigma}$ $(1/\rho + 1/\rho' = 1)$. Then $f \in L^p * L^\infty$ by Theorem 1 since $1/\rho' > 1/p'$ $(1/p + 1/p' = 1)$. But $f \notin M_r^{\sigma}$ for all $r \geqq 1$ because $\hat{f} \notin l^{\sigma}$. Thus this fact together with the usual duality argument shows that bounds of θ cannot be improved.

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