3. *De Rham Cohomologies and Stratifications*

Complex Analytic de Rham Cohomology. III

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The importance of the idea of stratifying varieties in the study of algebraic and analytic varieties is well known. The investigation of stratification of varieties would involve basically the following steps*):

(1) To stratify varieties so that each stratum as well as the relations among the strata, e.g., incidence relation , ⋯, are of simple (or typical) forms.

(2) To obtain results of desired nature for each stratum or each series of strata, etc. with respect to a fixed stratification for given varieties.

(3) To piece together results from the step (2) in order to obtain results of a desired sort for given varieties and subvarieties, ⋯.

The steps {(1), (2)} and (3) might be reasonably called, respectively, localization steps (for given global problems) and globalization steps (to be applied to local results).

In [5], [7] we investigated certain quantitative properties of real analytic varieties. Results of [5] are used in our study of the complex analytic de Rham cohomology. Our investigations in [7] are carried out using steps (1), (2) and (3). Exact sequences of Mayer-Vietoris type are used repeatedly in our globalization steps. The basis of our arguments used in the globalization steps is algebraic in nature.

The main purpose of the present note is to introduce the notion of *cochain complex with incidence relations* (C. C. I.) *for a prestratified space.* (See n.1. and n.2. below.) The arguments used in the study of C. C. I. are generalizations, as well as abstractions, of those in [7]. When C. C. I.'s are related to de Rham cohomologies of certain types, the arguments applicable to C. C. I.'s in general clarify relations between 'local' and 'global' [1] data in the de Rham cohomologies in question. Actually the author's hope in introducing the notion of C. C. I. is to clarify relations between 'local' and 'global' data in de Rham

*) See R. Thom [8], H. Whitney [9]. The author learned the theories of stratifications in connection with his proposed approach to Complex analytic de Rham cohomology. (Cf. [4], [5].)

1) The terms 'local' and 'global' in this note should be understood in the sense explained at the beginning of this note.

cohomologies of various types.

The cotents of this note are preliminary in nature. However, the arguments applicable to C. C. I. in general are indispensable in [5] and have certain theoretically pleasant aspects.

n.1. Prestratifications. Let X be a topological space. By a prestratification[2] $\mathcal{S}$ of X we mean a collection $\mathcal{S}=\{S_\lambda\}_{\lambda\in\Lambda}$ of subsets S_λ's of X satisfying the following conditions.

$(1.1)_1$ X is the disjoint union of S_λ's in $\mathcal{S}$: $X=\bigcup S_\lambda$, $S_\lambda\in\mathcal{S}$.

$(1.1)_2$ For each stratum $S_\lambda\in\mathcal{S}$, the dimension: $\dim S_\lambda\in Z^+$ is given.

$(1.1)_3$ Frontier condition: For each stratum $S_\lambda\in\mathcal{S}$, fron $(S_\lambda)=\bar{S}_\lambda-S_\lambda$ is the disjoint union of lower dimensional strata of $\mathcal{S}$.

To substantially simplify notations in later arguments we assume that

$(1.1)_4$ $\mathcal{S}$ is a finite set.[3]

A pair $(X,\mathcal{S})$ consisting of a topological space and its prestratification $\mathcal{S}$ will be called a *prestratified space*. Let $(X,\mathcal{S})$ be a prestratified space. For $\mathcal{T}\subset\mathcal{S}$, let $|\mathcal{T}|$ denote the support of $\mathcal{T}$: $|\mathcal{T}|=\bigcup S_\lambda$, $S_\lambda\in\mathcal{T}$. Moreover, for $\mathcal{T}'\subset\mathcal{T}\subset\mathcal{S}$, $\overline{|\mathcal{T}'|}_{\mathcal{T}}$ denotes the closure of $|\mathcal{T}'|$ in $|\mathcal{T}|$. We list certain notations used below. For $\mathcal{T}\subset\mathcal{S}$, define $\mathcal{T}_C$, $\mathcal{T}_m$ by

$$\mathcal{T}_C=\{\mathcal{T}'\subset\mathcal{T}:\overline{|\mathcal{T}'|}_{\mathcal{T}}-\mathcal{T}'=\phi \text{ or closed}\},$$
$$\mathcal{T}_m=\{S_\lambda\in\mathcal{T}: l(S_\lambda)\leqq m\}.^{4)}$$

Moreover, let $\mathcal{T}_0$ denote the collection of series of strata in $\mathcal{T}$:

$$\mathcal{T}_0=\{S_{\lambda_1,\cdots,\lambda_t}=S_{\lambda_1}<\cdots<S_{\lambda_t}: S_{\lambda_j}\in\mathcal{T}\ (j=1,\cdots,t)\}.$$

In the above $S_{\lambda_1}<S_{\lambda_2},\cdots$ means that $S_{\lambda_1}\subset$ fron $(S_{\lambda_2}),\cdots$. For $\mathcal{T}'\subset\mathcal{T}$ and a series $S_{\lambda_1,\cdots,\lambda_t}$, $S_{\lambda_j}\in\mathcal{T}'$, let $\mathcal{T}'_m(S_{\lambda_1,\cdots,\lambda_t})$ denote the intersection $\mathcal{T}'_m\cap\{S\in\mathcal{T}': S_{\lambda_t}<S\}$. For $\mathcal{T}\subset\mathcal{S}$ define $\mathcal{T}_{0C}$ by

$$\mathcal{T}_{0C}=\{\mathcal{T}'_m(S_{\lambda_1,\cdots,\lambda_t}):\mathcal{T}'\subset\mathcal{T},\ S_{\lambda_j}\in\mathcal{T}',\ j=1,\cdots,t\}.$$

Then one easily derives the following fact.

(1.2) If $\mathcal{T}\in\mathcal{S}_C$, then $\mathcal{T}_m\in\mathcal{S}_C$ and $\mathcal{T}_C,\mathcal{T}_{0C}\subset\mathcal{S}_C$. Moreover, if $\mathcal{T}_1$, $\mathcal{T}_2\in\mathcal{S}_C$ satisfy the relation $\mathcal{T}_1\vee\mathcal{T}_2$, then $\mathcal{T}_1\cup\mathcal{T}_2\in\mathcal{S}_C$.

Here $\mathcal{T}_1\vee\mathcal{T}_2$ if, for any $S_{\lambda_i}\in\mathcal{T}_i$ $(i=1,2)$, $S_{\lambda_1}\not< S_{\lambda_2}$, $S_{\lambda_1}\not> S_{\lambda_2}$.

n.2. Cochain complex with incidence relation (C. C. I.). Let $\mathcal{R}$ be a noetherian ring, and let $(X,\mathcal{S})$ be a prestratified space. Moreover, let $\mathcal{C}(\mathcal{S})$ be a collection of assignments $\{\mathcal{C}'(\mathcal{S}),\mathcal{E}(\mathcal{S}),\mathcal{E}'(\mathcal{S})\}$ of the following forms: $\mathcal{C}'(\mathcal{S}):\mathcal{T}\in\mathcal{S}_C\to\mathcal{C}'(\mathcal{T})$, $\mathcal{E}(\mathcal{S}):\mathcal{U}\in\mathcal{S}_0\to\mathcal{E}(\mathcal{U})$ and $\mathcal{E}'(\mathcal{S}):\mathcal{U}\in\mathcal{S}_{0C}\to\mathcal{E}'(\mathcal{U})$. Here $\mathcal{C}'(\mathcal{T})$, $\mathcal{E}(\mathcal{U})$ and $\mathcal{E}'(\mathcal{U})$ are $\mathcal{R}$-cochain complexes. The collection $\mathcal{C}(\mathcal{S})=\{\mathcal{C}'(\mathcal{S}),\mathcal{E}(\mathcal{S}),\mathcal{E}'(\mathcal{S})\}$ as above is called

2) For definitions of stratifications and prestratifications, see J. Mather [2], R. Thom [8]. Our definition of prestratifications is, for technical reasons, not the same as in [2], [8].

3) For the case where $\mathcal{S}$ is locally finite (see [6]).

4) For $S\in\mathcal{S}$, the length $l(S)$ can be defined in an obvious manner (see [6]).

a *cochain complex with incidence relations attached to* $(X, \mathcal{S})$ (C. C. I. attached to $(X, \mathcal{S})$) if $\mathcal{C}(\mathcal{S})$ is equipped with isomorphisms $i(S)$, $i(\mathcal{T}_1, \mathcal{T}_2)$, $i(\mathcal{U}_1)$ and homomorphisms[5] $h_k(\mathcal{T}_m)$, $h_k(\mathcal{U}_m)$ of the following forms.

$(1.3)_1$ $i(S): 0 \to \mathcal{C}'(S) \to \mathcal{E}(S) \to 0$, $S \in \mathcal{S}$.

$(1.3)_2$ $i(\mathcal{T}_1, \mathcal{T}_2): 0 \to \mathcal{C}'(\mathcal{T}_1 \cup \mathcal{T}_2) \to \mathcal{C}'(\mathcal{T}_1) \oplus \mathcal{C}'(\mathcal{T}_2) \to 0$, $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{S}_C$ and $\mathcal{T}_1 \vee \mathcal{T}_2$.

$(1.3)_2'$ $i(\mathcal{U}_1): 0 \to \mathcal{E}'(\mathcal{U}_1) \to \oplus_{\lambda_{t+1}} \mathcal{E}(S_{\lambda_1, \dots, \lambda_{t+1}}) \to 0$, where $\mathcal{U}_1 = \mathcal{T}_1(S_{\lambda_1, \dots, \lambda_t})$ with $\mathcal{T} \in \mathcal{S}_C$ and $S_{\lambda_j} \in \mathcal{T}$. Moreover, $S_{\lambda_{t+1}} \in \mathcal{U}_1$.

$(1.3)_3$ $0 \to \mathcal{C}'(\mathcal{T}_{m+1}) \xrightarrow{h_1(\mathcal{T}_m)} \mathcal{C}'(\mathcal{T}_m) \oplus \{\oplus_{\lambda m+1} \mathcal{C}'(S_{\lambda m+1})\} \xrightarrow{h_2(\mathcal{T}_m)}$ $\oplus_{\lambda m+1} \mathcal{E}'(\mathcal{T}_m(S_{\lambda m+1})) \to 0$, where $\mathcal{T} \in \mathcal{S}_C$ and $S_{\lambda m+1} \in \mathcal{T}_{m+1} - \mathcal{T}_m$.

$(1.3)_3'$ $0 \to \mathcal{C}'(\mathcal{U}_{m+1}) \xrightarrow{h_1(\mathcal{U}_m)} \mathcal{E}'(\mathcal{U}_m) \oplus \{\oplus_{\lambda_{t+1}^{m+1}} \mathcal{E}(S_{\lambda_1, \dots, \lambda_{t+1}^{m+1}})\} \xrightarrow{h_2(\mathcal{U}_m)}$ $\oplus_{\lambda_{t+1}^{m+1}} \mathcal{E}(\mathcal{U}_m(S_{\lambda_{t+1}^{m+1}})) \to 0$, where $\mathcal{U}_m = \mathcal{T}_m(S_{\lambda_1, \dots, \lambda_t})$ and $\mathcal{U}_{m+1} = \mathcal{T}_{m+1}(S_{\lambda_1, \dots, \lambda_t})$. Moreover, $S_{\lambda_{t+1}^{m+1}} \in \mathcal{U}_{m+1} - \mathcal{U}_m$.

Postulated conditions of the existence of isomorphisms in $(1.3)_1$, $\{(1.3)_2, (1.3)_2'\}$ and homomorphisms in $(1.3)_3$, $(1.3)_3'$ will be called *'Identification condition'*, *'Disjoint condition'* and *'Incidence condition'* (*'Mayer-Vietoris condition'*) respectively. The collection of isomorphisms $i(S), \cdots$ and homomorphisms h_k's will be denoted by $\mathcal{K}(\mathcal{C}(\mathcal{S}))$. When we emphasize the role of $\mathcal{K}(\mathcal{C}(\mathcal{S}))$, we say that $\mathcal{C}(\mathcal{S})$ is $\mathcal{K}(\mathcal{C}(\mathcal{S}))$-C. C. I. Let $\mathcal{C}(\mathcal{S})$ and $\mathcal{K}(\mathcal{C}(\mathcal{S}))$ be as above. Then isomorphisms $i(S), \cdots$ and homomorphisms h_k's of cochain complexes induce corresponding isomorphisms $i^*(S), \cdots$ and homomorphisms h_k^*'s of cohomology groups naturally. The collection of $i^*(S), \cdots$ and h_k^*'s will be denoted by $\mathcal{K}^*(\mathcal{C}(\mathcal{S}))$.

Equivalences between C. C. I.'s. Let $(X, \mathcal{S})$ be a prestratified space, and let $\mathcal{C}^i(\mathcal{S})$ be $\mathcal{K}(\mathcal{C}^i(\mathcal{S}))$-C. C. I. $(i=1,2)$. Moreover, let α^*, β^* and β'^* be families of $\mathcal{R}$-homomorphisms of the following forms: $\alpha^* = \{\alpha^*(\mathcal{T}): H^*(\mathcal{C}'^1(\mathcal{T})) \to H^*(\mathcal{C}'^2(\mathcal{T}))$, $\mathcal{T} \in \mathcal{S}_C\}$, $\beta^* = \{\beta^*(\mathcal{U}): H^*(\mathcal{E}^1(\mathcal{U})) \to H^*(\mathcal{E}^2(\mathcal{U}))$, $\mathcal{U} \in \mathcal{S}_0)$ and $\beta'^* = \{\beta'^*(\mathcal{U}): H^*(\mathcal{E}'^1(\mathcal{U})) \to H^*(\mathcal{E}'^2(\mathcal{U}))$, $\mathcal{U} \in \mathcal{S}_{0C}\}$. Then we can define the notion of *commutativity* of $\{\alpha^*, \beta^*, \beta'^*\}$ with $\mathcal{K}(\mathcal{C}^i(\mathcal{S}))$ in an obvious manner ([6]). We say that $\mathcal{C}^i(\mathcal{S})$ $(i=1,2)$ are $\{\alpha^*, \beta^*, \beta'^*\}$*-equivalent* if (i) $\{\alpha^*, \beta^*, \beta'^*\}$ commute with $\mathcal{K}(\mathcal{C}^i(\mathcal{S}))$ $(i=1,2)$ and if (ii) the homomorphism $\beta^*(\mathcal{U})$ is an isomorphism for any $\mathcal{U} \in \mathcal{S}_0$.

Now, in our investigations, there are reasons for regarding $\mathcal{C}'(\mathcal{T})$, $\mathcal{T} \in \mathcal{S}_C$ as 'global' data and $\mathcal{E}(\mathcal{U})$, $\mathcal{U} \in \mathcal{S}_0$ as 'local' data. The following lemma shows that certain properties of global data are derived from those of local data.

Lemma 1. *Let* $\mathcal{C}^i(\mathcal{S}) = \{\mathcal{C}'^i(\mathcal{S}), \mathcal{E}^i(\mathcal{S}), \mathcal{E}'^i(\mathcal{S})\}$ *be* $\mathcal{K}(\mathcal{C}^i(\mathcal{S}))$-C. C. I. $(i=0,1,2)$ *of a prestratified space* $(X, \mathcal{S})$.

(I) *If* $H^*(\mathcal{E}^0(\mathcal{U}))$ *is a finitely generated* $\mathcal{R}$*-module for each*

5) Isomorphisms and homomorphisms are those of R-cochain complexes.

$\mathcal{U} \in \mathcal{S}_0$, *then* $H^*(C'^0(\mathfrak{T}))$ *is so for each* $\mathfrak{T} \in \mathcal{S}_C$.

(II) *Let* $C^i(\mathcal{S})$ $(i=1,2)$ *be* $\{\alpha^*, \beta^*, \beta'^*\}$-*equivalent. Then* $\alpha^*(\mathfrak{T})$ *is an* $\mathfrak{R}$-*isomorphism for each* $\mathfrak{T} \in \mathcal{S}_C$. *Here* $\alpha^*, \beta^*, \beta'^*$ *are families of* $\mathfrak{R}$-*homomorphisms of the forms given in the beginning of n.2.*

For the proof of Lemma 1, see [6].

n.3. An exact sequence of Mayer-Vietoris type. Let K be an algebraically closed field of any characteristic. In n.3. every variety in question is assumed to be a reduced K-variety. Let A^n, V and D be an affine space of dimension n, a variety in A and a divisor in A, respectively, such that for each irreducible component V_j of V, $V_j \not\subset D$ and $V_j \cap D \neq \emptyset$. We denote by W the variety in A characterized by $|W| = |V| \cap |D|$. Now let $\mathfrak{O}$, $\mathfrak{J}_V$, $\mathfrak{J}_W$ and $\mathfrak{J}_D = \mathfrak{O}(h)$ denote respectively the ring $K[x_1, \cdots, x_n]$ and the ideals of V, W and D. The completions $\varprojlim_n \mathfrak{O}/\mathfrak{J}_V^n$ and $\varprojlim_n \mathfrak{O}/\mathfrak{J}_W^n$ are denoted by $\hat{\mathfrak{O}}^V$ and $\hat{\mathfrak{O}}^W$ respectively. We denote the localizations $\mathfrak{O}[h^{-1}]$ and $\hat{\mathfrak{O}}^W[h^{-1}]$ by $\mathfrak{O}[*D]$ and $\hat{\mathfrak{O}}^W[*D]$ respectively. Moreover, let $\hat{\mathfrak{O}}^{V-W}$ and $\hat{\mathfrak{O}}^{W,V-W}$ be respectively the completions defined by $\varprojlim_n \mathfrak{O}[*D]/(\mathfrak{O}[*D]\cdot\mathfrak{J}_V^n)$ and $\varprojlim_n \hat{\mathfrak{O}}^W[*D]/(\hat{\mathfrak{O}}^W[*D]\cdot\mathfrak{J}_V^n)$. In the above we regard $\mathfrak{J}_V$ as contained in $\mathfrak{O}[*D]$ and $\hat{\mathfrak{O}}^W[*D]$ in a natural fashion. For the graded ring Ω of K-differential forms over A, let $\hat{\Omega}^V, \hat{\Omega}^W, \hat{\Omega}^{W,V-W}$ respectively denote $\Omega \otimes_{\mathfrak{O}} \hat{\mathfrak{O}}^V, \cdots, \Omega \otimes_{\mathfrak{O}} \hat{\mathfrak{O}}^{W,V-W}$. Then we have:

Lemma 2. *For the rings* $\hat{\Omega}^V, \cdots, \hat{\Omega}^{W,V-W}$, *the exact sequence*

$$\text{(I)} \qquad 0 \to \hat{\Omega}^V \xrightarrow{\hat{\rho}^V_W \oplus \hat{\rho}^V_{V-W}} \hat{\Omega}^W \oplus \hat{\Omega}^{V-W} \xrightarrow{\hat{\rho}^W_{W,V-W} \oplus \hat{\rho}^{V-W}_{W,V-W}} \hat{\Omega}^{W,V-W} \to 0,$$

holds. In (I) *the continuous homomorphisms* $\hat{\rho}^V_W, \cdots, \hat{\rho}^{V-W}_{W,V-W}$ *are determined naturally from topologies of* $\hat{\Omega}^V, \cdots, \hat{\Omega}^{W,V-W}$.

For the proof of Lemma 2, see [6]. The exact sequence (I) relates cohomology groups $H^*(\hat{\Omega}^V), H^*(\hat{\Omega}^{V-W})$ and $H^*(\hat{\Omega}^{W,V-W})$ to the cohomology group $H^*(\hat{\Omega}^V)$. For a pair (V, W) of smooth varieties (defined over the complex number field $\boldsymbol{C}$), the idea of relating the cohomology groups of $W, V-W$ and $N(W)-W$[6] to that of V may be regarded as one of the basic ideas in the *classical* (*analytic*) *theories of residues.* (Cf. J. Leray [2], P. A. Griffith [1], $\cdots$). The sequence (I) might be regarded as a generalization in an algebraic direction of the idea explained above. Moreover, Lemma 2 enables us to attach the *algebraic de Rham* C. C. I. to the prestratified space $(V, \mathcal{S}=(W, V-W))$ in a natural manner.

Remarks about results untouched here. In this note, we have spent several pages explaining ideas used in defining C. C. I.'s. For arguments on C. C. I.'s untouched here, see [6]. In particular, [6] contains examples of C. C. I.'s such as the *singular, the* C^∞-*de Rham,*

6) $N(W)$ is a suitable neighbourhood of W in V.

and the P. G. (polynomial growth) de Rham C. C. I.'s, as well as an application of the sequence (I).

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