24. *The Local Maximum Modulus Principle for Function Spaces*

By Yukio HIRASHITA*) and Junzo WADA**)

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The local maximum modulus principle for function algebras due to H. Rossi [5] is well-known. The purpose of this paper is to consider the principle for function spaces, more correctly speaking, for function systems. In § 1, for any function system $\mathcal{F}$, we define the $LMM(\mathcal{F})$-boundary which plays the same rôle as the Shilov boundary in the Rossi's principle. In §§ 2 and 3, properties of the $LMM(\mathcal{F})$-boundary and relations between the Rossi's principle and ours are discussed.

§ 1. **The *LMM*-boundary.** Let X be a compact Hausdorff space. For any subset S in X, $\dot{S}$ denotes the topological boundary of S, i.e., $\dot{S}=\bar{S}\backslash S^i$, where $\bar{S}$ and S^i are the closure and the interior of S in X respectively.

Let $\mathcal{F}$ be a family of complex-valued bounded continuous functions defined on subsets of X. We denote the domain of f by $D(f)$ $(f\in\mathcal{F})$. $\mathcal{F}$ is said to be a *function system* on X if $\mathcal{F}$ has the following properties:

(1) If $f, g\in\mathcal{F}$ and α, β are complex numbers, then $\alpha f+\beta g$ (defined on $D(f)\cap D(g)$) belongs to $\mathcal{F}$.

(2) $\mathcal{F}_X=\{f\in\mathcal{F}: D(f)=X\}$ separates points of X and contains constant functions.

Let $\mathcal{F}$ be a function system on X. We will say that a subset E of X satisfies the *$LMM(\mathcal{F})$-principle* if $\|f\|_{\dot{U}}=\|f\|_U$ for any open subset U in X with $U\cap E=\phi$ and for any $f\in\mathcal{F}$ with $D(f)\supset\bar{U}$, where $\|f\|_P=\sup_{x\in P}|f(x)|$ for any P ($\|f\|_\phi=0$ for the empty set ϕ).

We shall first show that there exists the smallest one F_0 among non-void[1] closed subsets which satisfy the $LMM(\mathcal{F})$-principle. Such set F_0 is called the *$LMM(\mathcal{F})$-boundary* and we write $F_0=LMM(\mathcal{F})$.

Theorem 1.1. *For any function system $\mathcal{F}$, there exists the $LMM(\mathcal{F})$-boundary.*

Proof. Let $\mathcal{P}=\{F_\lambda\}_{\lambda\in\Lambda}$[2] be the family of all (non-void) closed subsets in X which satisfy the $LMM(\mathcal{P})$-principle. We define a partial order $\succ$ in Λ as follows: $\lambda\succ\mu$ if and only if $F_\lambda\supset F_\mu$. It is not hard to

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*) Kyushu University, Fukuoka.
**) Waseda University, Tokyo.

1) The empty set ϕ does not satisfy the $LMM(\mathcal{F})$-principle.
2) $\mathcal{P}$ is non-void, because $\mathcal{P}\ni X$.

see that any totally ordered subset of Λ has a lower bound. Hence Zorn's lemma guarantees that $\mathcal{P}$ has a minimal one F_0. To complete our proof we verify that F_0 is the smallest one of $\mathcal{P}$. The proof is similar to Bear's [1]. Let a closed subset B have the $LMM(\mathcal{F})$-principle. Then we shall show that $F_0 \subset B$. Suppose that $F_0 \not\subset B$, then there exist $x_0 \in F_0 \backslash B$, and a non-void open subset V ($\ni x_0$) with $V \cap B = \phi$. Since $\mathcal{F}_X$ separates points in X, the ordinary topology on X coincides with the weak topology on X with respect to $\mathcal{F}_X$. From this we can assume that V is of the form $\{x \in X : |f_i(x) - f_i(x_0)| < \varepsilon\}$, where $f_i \in \mathcal{F}_X$ $(i=1,2,\cdots,n)$ and $\varepsilon > 0$. By setting $g_i = f_i - f_i(x_0)$ ($\in \mathcal{F}_X$), we have $V = \{x \in X : |g_i(x)| < \varepsilon,\ i=1,2,\cdots,n\}$. If $T = F_0 \backslash V$, by the minimality of F_0, T fails to satisfy the principle. Hence there exist an open subset U and $f \in \mathcal{F}$ such that $U \cap T = \phi$, $D(f) \supset \overline{U}$ and $\|f\|_{\dot{U}} < \|f\|_U$. We can here choose a sufficiently large number m such that $g = mf$ satisfies the following:

$$\|g_1\|_U + \|g_2\|_U + \cdots + \|g_n\|_U + \|g\|_{\dot{U}} < \|g\|_U.$$

Now let α be any complex number with $|\alpha| = 1$. Then for any $x \in U \cap V$ and any $k \in \{1, 2, \cdots, n\}$

$$|g(x) + \alpha g_k(x)| \leqq |g(x)| + |g_k(x)| < \|g\|_U + \varepsilon.$$

If $x \in \dot{U}$, then

$$|g(x) + \alpha g_k(x)| \leqq \|g\|_{\dot{U}} + \|g_k\|_U < \|g\|_U.$$

If we set $W = U \backslash F_0$, then $\dot{W} \subset \dot{U} \cup \{U \cap V\}$ and $D(g + \alpha g_k) \supset \overline{W}$, and by two inequalities above,

$$\|g + \alpha g_k\|_{\dot{W}} < \|g\|_U + \varepsilon.$$

Since $W \cap F_0 = \phi$, by the $LMM(\mathcal{F})$-principle,

$$\|g + \alpha g_k\|_W = \|g + \alpha g_k\|_{\dot{W}} < \|g\|_U + \varepsilon.$$

It follows that $\|g + \alpha g_k\|_U < \|g\|_U + \varepsilon$, because $\overline{U} = \dot{U} \cup U = \dot{U} \cup (U \backslash F_0) \cup (U \cap F_0) = \dot{U} \cup W \cup (U \cap F_0) \subset \dot{U} \cup W \cup (U \cap V)$.

We here take any $t \in M_g = \{x \in \overline{U} : |g(x)| = \|g\|_U\}$, then there exists an α ($|\alpha| = 1$) such that

$$|g(t) + \alpha g_k(t)| = |g(t)| + |g_k(t)|.$$

Hence we have

$$\begin{aligned} \|g\|_U + |g_k(t)| &= |g(t)| + |g_k(t)| = |g(t) + \alpha g_k(t)| \\ &\leqq \|g + \alpha g_k\|_U < \|g\|_U + \varepsilon. \end{aligned}$$

It implies that $|g_k(t)| < \varepsilon$ $(k=1,2,\cdots,n)$, and so $M_g \subset V$. Since $M_g \subset U$, $M_g \subset U \cap V \equiv S$. It follows that $\|g\|_{\dot{S}} < \|g\|_S$ and $S \cap B \subset V \cap B = \phi$. This shows that B fails to satisfy the $LMM(\mathcal{F})$-principle. It concludes that F_0 is the $LMM(\mathcal{F})$-boundary.

§ 2. **The $LMM(\mathcal{F})$-boundary and the Shilov boundary.** A linear subspace A of $C(X)$ is said to be a function space on X if A separates points of X and contains constant functions.

Let A be a function space on X. A function f defined on S ($\subset X$)

is said to be (A-) holomorphic on S if for any $x \in S$ there exists a neighborhood V of x in X such that f can be approximated uniformly on $S \cap V$ by functions in A. We denote the set of all holomorphic functions on S by $\mathcal{H}_A(S)$. Let $\mathcal{H}'_A(S)$ denote the set of all functions on S which can be approximated uniformly on S by functions of A.

For a function space A on X, the following three are function systems on X: (1) $\mathcal{F}(A)=A$, (2) $\mathcal{F}(\mathcal{H}'_A)=\bigcup_{S\subset X}\mathcal{H}'_A(S)$ and (3) $\mathcal{F}(\mathcal{H}_A)=\bigcup_{S\subset X}\mathcal{H}_A(S)$.

Theorem 2.1. $\partial_A \subset LMM(\mathcal{F}(A)) = LMM(\mathcal{F}(\mathcal{H}'_A)) \subset LMM(\mathcal{F}(\mathcal{H}_A))$, *where ∂_A denotes the Shilov boundary for A.*

Proof. It suffices to prove only that $\partial_A \subset LMM(\mathcal{F}(A))$. We set $U=X\backslash LMM(\mathcal{F}(A))$. Then for any $f\in A$, $\|f\|_{\dot{U}}=\|f\|_U$. Since $\dot{U}\subset LMM(\mathcal{F}(A))$, we have $\|f\|_X=\max\{\|f\|_{X\backslash U}, \|f\|_{\dot{U}}\}=\|f\|_{LMM(\mathcal{F}(A))}$. This shows $\partial_A\subset LMM(\mathcal{F}(A))$.

A similar result as Corollary 2.3 of Rickart [4] can be obtained as follows.

Theorem 2.2. *If $U\cap LMM(\mathcal{F}(\mathcal{H}_A))=\phi$ for a non-void open subset U in X and $h\in\mathcal{H}_A(U)$, then there exists $\delta\in\dot{U}$ such that $\|h\|_U=\|h\|_{U\cap V}$ for any open neighborhood V of δ.*

§ 3. Singular points. Let A be a function space on X and $\varphi: X \to A^*$ denote the canonical mapping from X to the dual space A^* with weak*-topology. We can identify X and $\varphi(X)$ in the usual sense: $\langle\varphi(x), f\rangle=f(x)$ for $x\in X, f\in A$. For $S\subset X$, $\widehat{\varphi(S)}$ denotes the (w^*-) closed convex hull of $\varphi(S)$. We see that $\widehat{\varphi(X)}$ equals the state space $\{L\in A^*: L(1)=1=\|L\|\}$ (cf. [3]). We write $\hat{S}$ in the place of $\widehat{\varphi(S)}$. A point $x\in X$ is said to be *singular* if there exists an open neighborhood V of x in X such that $x\in \mathrm{ex}\,\hat{V}$, where $\mathrm{ex}\,\hat{V}$ denotes the set of all extreme points of $\hat{V}$. We denote by S_A the set of all singular points.

Theorem 3.1. *$LMM(\mathcal{F}(A))$ is equal to the closure $\bar{S}_A$ of S_A.*

Proof. (1) If $LMM(\mathcal{F}(A))\subsetneq \bar{S}_A$, then $\bar{S}_A$ fails to have the $LMM(\mathcal{F}(A))$-principle. Hence there are an open subset U and an $f\in A$ such that $U\cap\bar{S}_A=\phi$ and $\|f\|_{\dot{U}}<\|f\|_U$. Since f can be considered as a continuous affine function on $\hat{U}(\subset A^*)$, there exists $x_0\in\mathrm{ex}\,\hat{U}$ such that $|f(x_0)|=\|f\|_{\hat{U}}=\|f\|_U$. Since $x_0\in\bar{U}$ (cf. [3]) and $\|f\|_{\dot{U}}<\|f\|_U$, we have $x_0\notin\dot{U}$, and so $x_0\in U$. It implies $x_0\in S_A$, which contradicts that $U\cap S_A=\phi$.

(2) If $S_A\subsetneq LMM(\mathcal{F}(A))$, we choose $x_0\in S_A\backslash LMM(\mathcal{F}(A))$. Then there exists an open subset U such that $U\ni x_0$ and $\mathrm{ex}\,\hat{U}\ni x_0$. Let $V=U\backslash LMM(\mathcal{F}(A))$, then $V\neq\phi$ and $V\cap LMM(\mathcal{F}(A))=\phi$. We can here show that $\dot{V}\subset F\equiv\dot{U}\cup\{\bar{U}\cap LMM(\mathcal{F}(A))\}$ and $F\not\ni x_0$. Now suppose that $x_0\in\hat{\dot{V}}$, then $x_0\in\hat{\dot{V}}\subset\hat{F}\subset\hat{\bar{U}}=\hat{U}$. Since $x_0\in\mathrm{ex}\,\hat{U}$, we have $x_0\in\mathrm{ex}\,\hat{F}$ and so $x_0\in F$. This contradiction shows $x_0\notin\hat{\dot{V}}$. Since $x_0\in\mathrm{ex}\,\hat{\bar{U}}$, there exists

an $f\in A$ such that $\|f\|_{\dot{V}}^{\wedge}<|f(x_0)|$ ([3]). From this,

$$\|f\|_{\dot{V}}=\|f\|_{\dot{V}}^{\wedge}<|f(x_0)|\leqq\|f\|_V.$$

This is a contradiction, because $V\cap LMM(\mathcal{F}(A))=\phi$.

When $\bar{S}_A=\partial_A$, we have

Theorem 3.2. *If $\bar{S}_A=\partial_A$, then $\partial_A=LMM(\mathcal{F}(\mathcal{H}_A))$.*

Proof. Since $\partial_A\subset LMM(\mathcal{F}(\mathcal{H}_A))$ by Theorem 2.1, we have to show only that $\partial_A\supset LMM(\mathcal{F}(\mathcal{H}_A))$. For any open subset U in X with $U\cap\partial_A=\phi$ and for any $h\in\mathcal{H}_A(\bar{U})$, B denotes the function space generated by $\{A\,|\,\bar{U},h\}$. Assume that $U\cap\delta_B\neq\phi$, where δ_B is the Choquet boundary for B. We choose $x_0\in U\cap\delta_B$. Then for any open subset $V\ni x_0$, there exists $f\in B$ such that $\|f\|_{\bar{U}\setminus V}<|f(x_0)|$. Since h is holomorphic, h is approximated uniformly by functions in A on some open subset $W(U\supset\bar{W}\supset W\ni x_0)$. Hence $\|f\|_{\bar{V}\setminus W}<|f(x_0)|$ for some $f\in B$. It follows that $f\,|\,\bar{W}\in\mathcal{H}_A'(\bar{W})$ and $\|f\|_{\dot{W}}\leqq\|f\|_{\bar{U}\setminus W}<|f(x_0)|\leqq\|f\|_W$. Since $W\cap\partial_A=\phi$ and $\partial_A=\bar{S}_A=LMM(\mathcal{F}(\mathcal{H}_A'))$ by Theorems 2.1 and 3.1, this is a contradiction. This shows $U\cap\delta_B=\phi$, that is, $\delta_B\subset\dot{U}$. It implies that $\|h\|_U=\|h\|_{\delta_B}\leqq\|h\|_{\dot{U}}\leqq\|h\|_U$, and so ∂_A satisfies the $LMM(\mathcal{F}(\mathcal{H}_A))$-principle. Thus the theorem is proved.

Now, let $x_0\in S_A$. Then we see that there exists an open subset $W(\ni x_0)$ in X which has the following property: for any open neighborhood U of x_0 with $U\subset W$, there is an $f\in A$ such that $U\supset\{x\in\bar{W}: f(x)=\|f\|_W\}$. By this fact and the local peak set theorem ([5] or [2], p. 91), the Rossi's principle can be written as follows.

Theorem 3.3. *Let A be a function algebra on the maximal ideal space M_A. Then $\partial_A=\bar{S}_A$.*

References

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