34. *Unipotent Elements and Characters of Finite Chevalley Groups*

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Let $\mathfrak{G}$ be a connected semisimple linear algebraic group defined over an algebraically closed field K of characteristic $p>0$, and σ a surjective endomorphism of $\mathfrak{G}$ such that the group $\mathfrak{G}_\sigma$ of fixed points is finite. A finite group $G=\mathfrak{G}_\sigma$ obtained in this manner is called a finite Chevalley group. The purpose of this note is to announce some results concerning unipotent elements and (complex) characters of a finite Chevalley group $G=\mathfrak{G}_\sigma$. The proof is given in the author's forthcoming paper [8]. After the paper [8] was submitted to Osaka Journal of Mathematics, the author received two preprints [9] and [10], in which Theorems II, IV and V below are proved independently.

1. Let (G,B,N,S) be a Tits system (or BN-pair) associated to a finite Chevalley group G. We denote by W its Weyl group. Let G^1 be the set of unipotent elements (or p-elements) of G, and U the p-Sylow subgroup of G contained in B. For a finite set A, $|A|$ denotes the number of its elements.

Theorem I. *Let w be an arbitrary element of W, and w_S the element of W of maximal length. Then the number of unipotent elements of G contained in the double coset BwB is $|BwB\cap w_S U w_S^{-1}|\,|U|$.*

Corollary. $|G^1|=|U|^2$.

Remarks. (a) In [8], we will prove a formula for the number of unipotent elements contained in $BwB\cap P$, where P is an arbitrary parabolic subgroup of G. Theorem I above is a special case of this formula.

(b) The above corollary is originally proved by R. Steinberg [7].

2. An element x of $\mathfrak{G}$ is called regular if $\dim Z_{\mathfrak{G}}(x)=\operatorname{rank}\mathfrak{G}$, where $Z_{\mathfrak{G}}(x)$ is the centralizer of x in $\mathfrak{G}$. In [6], Steinberg proved the existence of regular unipotent elements of $\mathfrak{G}$. For example, if $\mathfrak{G}=SL_n$, a unipotent element of $\mathfrak{G}$ is regular if and only if its Jordan normal form consists of a single block. Below, we call an element of $G=\mathfrak{G}_\sigma$ regular if it is regular as an element of $\mathfrak{G}$.

Theorem II. *Assume that the characteristic p is good* (*see* [1; *Part E*]) *for $\mathfrak{G}$. Let g be an arbitrary element of $G=\mathfrak{G}_\sigma$, and C a regular unipotent conjugacy class of G. Then the number $|Bg\cap C|$ depends neither on g nor C.*

3. For a subgroup H of G, 1_H denotes the trivial character of H. If θ is a character of H, $i[\theta|H\to G]$ denotes the character of G induced from θ. Let P be a parabolic subgroup of G. Since P is a finite group with a BN-pair, there is an irreducible character ξ_P of P called the Steinberg character (see [2]).

Theorem III. *Let P and ξ_P be as above. Let χ be an irreducible character of G contained in $i[1_B|B\to G]$. Then*

$$\sum_{u\in G^1} \chi(u) i[1_P|P\to G](u) = \sum_{u\in G^1} \hat{\chi}(u) i[\xi_P|P\to G](u),$$

where $\hat{\chi}$ is the "dual character" ([5]) of χ defined using Goldman's involutory automorphism ([4]) of the Hecke algebra $H_C(G,B)$.

Corollary. $\sum_{u\in G^1} \chi(u) = |U|\hat{\chi}(1)$.

4. Theorem IV. *Assume that $\mathfrak{G}$ is adjoint and p is good for $\mathfrak{G}$. Let χ be an irreducible cuspidal character of $G=\mathfrak{G}_\sigma$, and u a regular unipotent element of G. Then $\chi(u)=\pm 1$ if χ is contained in the character induced from a linear character of U in "general position" (in the sense of Gel'fand and Graev [3]), and $\chi(u)=0$ otherwise.*

5. Theorem V. *Assume that p is good for $\mathfrak{G}$. Let χ be an irreducible character of $G=\mathfrak{G}_\sigma$ contained in $i[1_B|B\to G]$. Let u be a regular unipotent element of G. Then*

$$\chi(u) = \begin{cases} 1 & \text{if } \chi = 1_G, \\ 0 & \text{if } \chi \neq 1_G. \end{cases}$$

Using an elementary lemma, Theorem II and Theorem V can be translated into each other. The author does not know whether these two theorems hold in any characteristic $p>0$ or not. But, at least, the following weaker forms of II and V hold without restriction on $p>0$.

Theorem II′. *The number of regular unipotent elements contained in each coset Bg ($g\in G$) does not depend on g.*

Theorem V′. *Let χ be as in Theorem V, and G^1_r the set of regular unipotent elements of G. Then*

$$\sum_{u\in G^1_r} \chi(u) = \begin{cases} |G^1_r| & \text{if } \chi = 1_G, \\ 0 & \text{if } \chi \neq 1_G. \end{cases}$$

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