33. *Local Solvability of a Class of Partial Differential Equations with Multiple Characteristics*

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§ 1. Introduction. The present paper is concerned with local solvability for the following type of operators with C^∞ coefficients

$$L(x;\partial_x)=P(x;\partial_x)+Q(x;\partial_x)+R(x;\partial_x) \qquad (x\in \boldsymbol{R}^n),$$

where $P(x;\partial_x)$, $Q(x;\partial_x)$, $R(x;\partial_x)$ are the principal part of order s, the homogeneous part of order $s-1$, and the part of order $s-2$, respectively. When $P(x;\partial_x)$ is of principal type, L. Nirenberg-F. Treves [3] and R. Beals-G. Fefferman [1] established the necessary and sufficient condition for local solvability. On the other hand, when $P(x;\partial_x)$ has double characteristics, a necessary condition is given by F. Cardoso-F. Treves [2]. In that paper they pointed out that the subprincipal part of $L(x;\partial_x)$ plays an important role.

In this paper, we give a sufficient condition under some hypotheses not only for the principal part but for the subprincipal part. A forthcoming article will give a detailed proof. Let V_x be a neighbourhood of the origin in $\boldsymbol{R}^n_x$, and $\boldsymbol{S}^{n-1}_\xi$ be the unit sphere in $\boldsymbol{R}^n_\xi$. For the principal symbol $P(x;\xi)$, we assume that the characteristics of $P(x;\xi)$ have locally constant multiplicities in $V_x\times \boldsymbol{S}^{n-1}_\xi$. Under this assumption when we divide $J=\{(x,\xi)\in V_x\times S^{n-1}_\xi \mid P(x;\xi)=0\}$ into the connected components $\{J_k\}$, $P(x;\xi)$ vanishes of constant order m_k on J_k. Moreover, for simplicity, we assume that $P(x;\partial_x)$ has real coefficients.

§ 2. Statement of the theorem. Let us put

$$J^{(2)}=\{(x,\xi)\in J \mid \mathrm{grad}_\xi\, P(x;\xi)=0\}$$

and divide it into the connected components $\{J^{(2)}_k\}$. For the subprincipal symbol

$$\Pi(x;\xi)=Q(x;\xi)-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial_{x_j}\partial_{\xi_j}}P(x;\xi),$$

we assume that on each $J^{(2)}_k$, $\Pi(x;\xi)$ satisfies one of the following conditions:

(A) $\mathrm{Re}\,\Pi(x;\xi)\neq 0$ on $J^{(2)}_k$.

(B) $\Pi(x;\xi)\equiv 0$ on $J^{(2)}_k$ and if $m_k\geqslant 3$ moreover
$\mathrm{grad}_\xi\,\mathrm{Re}\,\Pi(x;\xi)\neq 0$ on $J^{(2)}_k$.

When the above assumptions are satisfied, we have the following proposition.

Proposition. *For arbitrary real number l, there exists a neigh-*

bourhood Ω of the origin in $\boldsymbol{R}^n_x$, such that

$$\|L^*u\|_{-l-s+2,\boldsymbol{R}^n} \geqslant c\,\|u\|_{-l,\boldsymbol{R}^n} \qquad \text{for all } u(x) \in \mathcal{D}(\Omega).$$

Especially when (A) *is always satisfied on* $J^{(2)}$, *we have*

$$\|L^*u\|_{-l-s+1,\boldsymbol{R}^n} \geqslant c\,\|u\|_{-l,\boldsymbol{R}^n} \qquad \text{for all } u(x) \in \mathcal{D}(\Omega).$$

The above proposition implies that the completion of $\mathcal{D}(\Omega)$ with the inner product $(L^*u, L^*v)_{-l-s+2,\boldsymbol{R}^n}$ [$(L^*u, L^*v)_{-l-s+1,\boldsymbol{R}^n}$, respectively] gives a Hilbert space $\mathfrak{H}$, which is contained $H_{\bar{\Omega}}^{-l}(\boldsymbol{R}^n)$ as a dense subspace. Therefore by the relation $\mathfrak{H}' \supset H^l(\Omega)$ and Riesz' Theorem, we have

Theorem. *Under the same assumptions as in Proposition, for each real number l, there exists a neighbourhood Ω of the origin in $\boldsymbol{R}^n_x$ which satisfies the following:*

For each $f(x)$ in $H^l(\Omega)$ there exists a solution $v(x)$ of $Lv=f$ in $H^{l+s-2}(\Omega)$. Moreover if (A) *is satisfied on all over $J^{(2)}$, we can take $v(x)$ in $H^{l+s-1}(\Omega)$.*

§ 3. Outline of the proof of Proposition. 1. Localization and modification of $L(x;D)$. $L^*(x;\partial_x)$ satisfies the same conditions as $L(x;\partial_x)$. Therefore, for simplicity, we consider $L(x;\partial_x)$ instead of $L^*(x;\partial_x)$.

We modify the coefficients of $L(x;D)$ out of a small neighbourhood of the origin in $\boldsymbol{R}^n_x$ in order to make the oscillation in x of $L(x;\xi)$ sufficiently small. Next, let us localize $L(x;D)$ in $\boldsymbol{R}^n_\xi$. We take an element $\alpha(\xi)$ in $\mathcal{D}(\boldsymbol{S}^{n-1}_\xi)$ whose support is sufficiently small and intersects at most a component of $\{J_k\}$. For $\xi \in \boldsymbol{R}^n - \{0\}$, let $\alpha(\xi) = \alpha(\xi/|\xi|)$. We use Λ whose symbol is $(|\xi|^2+1)^{1/2}$, and denote l instead of $l+s-2$ [$l+s-1$, respectively].

$$\begin{aligned}\alpha(D)\Lambda^{-l}L(x;D)u = {} & P(x;D)(\alpha\Lambda^{-l}u) - iQ(x;D)(\alpha\Lambda^{-l}u)\\ & -i\,\mathrm{grad}_\xi\,\Lambda^{-l}\cdot\mathrm{grad}_x\,P(x;D)(\alpha u)\\ & -i\,\mathrm{grad}_\xi\,\alpha(D)\cdot\mathrm{grad}_x\,P(x;D)\Lambda^{-l}u\\ & -i\tilde{R}(x;D),\end{aligned}$$

where $\tilde{R}(x;D)$ is of order $s-2$. In the case where supp $[\alpha]$ does not intersect on J, easily we have

$$\|\alpha\Lambda^{-l}Lu\| \geqslant c\,\|\alpha u\|_{s-l} - C\,\|u\|_{s-l-2} - C(h)\,\|u\|_{s-l-3}.$$

And in the case where supp $[\alpha]$ intersects on $J\backslash J^{(2)}$, for $\Omega = B_h$ (B_h is the ball with the radius h and the centre at the origin.), we have

$$\|\alpha\Lambda^{-l}Lu\| \geqslant \frac{c}{h}\,\|\alpha u\|_{s-l-1} - C\,\|u\|_{s-l-2} - C(h)\,\|u\|_{s-l-3}.$$

In the other cases, by a suitable rotation of the coordinates we can express

$$P(x;\xi) = a_0(x)(\xi_1 - \psi(x;\xi'))^m\Big\{\xi_1^{s-m} - \sum_{j=0}^{s-m-1} a_j(x;\xi')\xi_1^j\Big\},$$

where $a_0(x)$ and $P_0(x;\xi) \equiv \xi_1^{s-m} - \sum_{j=0}^{s-m-1} a_j(x;\xi')\xi_1^j$ do not vanish on $V_x \times \mathrm{supp}\,[\alpha]$, and where $\psi(x;\xi')$ and $a_j(x;\xi')$ $(0 \leqslant j \leqslant s-m-1$ and ξ'

$=(\xi_2, \cdots, \xi_n)$) are infinitely differentiable in x and holomorphic in ξ'. In case under the condition (B), applying the estimate under the condition (A) for all $u(x)$ in $\mathcal{D}(B_h)$, we can easily obtain the estimate

$$\|\alpha Lu\|_{-l} \geqslant \frac{c}{h} \|\alpha u\|_{s-l-2} - C\|u\|_{s-l-2} - C(h)\|u\|_{s-l-3}.$$

Therefore in this paper we only consider the case under the condition (A).

Since we can modify the symbols $\psi(x;\xi')$ and $a_j(x;\xi')$ $(0\leqslant j\leqslant s-m-1)$ out of supp$[\alpha]$, we can extend $\psi(x;\xi')$ and $a_j(x;\xi')$ suitably as the elements in $C^\infty(V_x \times (R^{n-1}_{\xi'}-\{0\}))$. We put $\Pi_0(x;D)=P_0(x;D)(D_1-\psi(x;D'))^m$ and $i\Pi_1(x;D)=\Pi_0(x;D)-P(x;D)+iQ(x;D)+i\,\mathrm{grad}_\xi \Lambda^{-l} \times \mathrm{grad}_x P(x;D)\Lambda^l$, where $D_1=i^{-1}(\partial/\partial_{x_1})$ and $D'=i^{-1}(\partial/\partial_{x'})$. Let us remark that

$$\Pi_1^0(x,\xi)\equiv \Pi(x;\xi) \qquad \text{modulo } (\xi_1-\psi(x;\xi'))^{m-1},$$

where Π_1^0 is the principal part of Π_1.

2. Reduction to a first order system. Let $\tilde{u}=\alpha(D)\Lambda^{-l}u$, $\tilde{u}_j=\alpha_{\xi_j}(D)\Lambda^{-l+1}u$ and Λ_0 be a pseudo-differential operator with the symbol $|\xi'|=(\sum_{j=2}^n \xi_j^2)^{1/2}$. Put

$$u_k=\begin{cases}(\Lambda_0+1)^{s-k}(D_1-\psi(x;D'))^{k-1}\tilde{u} & (1\leqslant k\leqslant m+1),\\ (\Lambda_0+1)^{s-k}D_1^{k-m-1}(D_1-\psi(x;D'))^m\tilde{u} & (m+2\leqslant k\leqslant s),\end{cases}$$

$$u_{jk}=\begin{cases}(\Lambda_0+1)^{s-k}(D_1-\psi(x;D'))^{k-1}\tilde{u}_j & (1\leqslant k\leqslant m+1),\\ (\Lambda_0+1)^{s-k}D_1^{k-m-1}(D_1-\psi(x;D'))^m\tilde{u}_j & (m+2\leqslant k\leqslant s),\end{cases}$$

$\Pi_1\tilde{u}=\sum_{k=1}^s b_k(x;D')u_k$, $U=(u_k)$ and $U_j=(u_{jk})$. Then we have

$$\|\alpha Lu\|_{-l}=\left\|D_1U-(H+B+G)U-\sum_{j=1}^n K_jU_j+U'\right\|=\|L_0U\|,$$

where

$$H=\left(\begin{array}{cccc|cccc}\psi & \Lambda_0 & & & & & & \\ & \ddots & \ddots & & & 0 & & \\ & & \ddots & \Lambda_0 & & & & \\ & & & \psi & \Lambda_0 & & & \\ \hline & & & & 0 & \Lambda_0 & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & 0 & \Lambda_0 \\ ib_1 & \cdots & \cdots & ib_m & a_0+ib_{m+1} & \cdots & \cdots & a_{s-m-1}+ib_s\end{array}\right),$$

$$B=\left(\begin{array}{cccc|cccc}c_1 & 1 & & & & & & \\ & \ddots & \ddots & & & 0 & & \\ & & \ddots & 1 & & & & \\ & & & c_m & 1 & & & \\ \hline & & & & 0 & 1 & & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & 0\end{array}\right),$$

c_j being of order 0 $(1\leqslant j\leqslant m)$, G is of order -1, $K_j=\begin{pmatrix} & 0 & \\ 0\cdots 0 & k_{jm}\cdots k_{js}\end{pmatrix}$

$(1 \leqslant j \leqslant n,\ k_{jk}$ being of order 0) and $\|U'\| \leqslant c\|u\|_{-l+s-2}$.

3. Jordan's canonical form of H. The eigenvalues μ_j $(1 \leqslant j \leqslant m)$ corresponding to ψ are expanded in the sense of Puiseux by $|\xi'|^{-1/m}$ and distinct. For the rest ones, we consider the eigenvalues $\lambda_j(|\xi'|)$ of

$$H_1(x\,;\xi')=\left(\begin{array}{ccc|cccc} \psi(x\,;\xi') & |\xi'| & & & & 0 & \\ & \ddots & \ddots |\xi'| & & & & \\ & & \psi(x\,;\xi') & |\xi'| & & & \\ \hline & & & 0 & |\xi'| & & \\ & 0 & & & \ddots & \ddots & \\ & & & & & 0 & |\xi'| \\ & & & a_0(0\,;\xi'_0)\,|\xi'|/|\xi'_0| & \cdots & & a_{s-m-1}(0\,;\xi'_0)\,|\xi'|/|\xi'_0| \end{array}\right),$$

where ξ_0 is a fixed point in supp $[\alpha] \cap S_\xi^{n-1}$. On $S_{\xi'}^{n-2}$, λ_j are constant roots with mutiplicities r_j $(1 \leqslant j \leqslant p$ and $\sum_{j=1}^{p} r_j = s-m)$. Since we can have exact expressions of the eigen-vectors V_k corresponding to μ_k $(1 \leqslant k \leqslant m)$ and of root vectors V_{jk} corresponding to λ_j $(1 \leqslant j \leqslant p,\ 1 \leqslant k \leqslant r_j)$. We put $\mathcal{N} = {}^t(V_1, \cdots, V_m, V_{11}, \cdots, V_{1r_1}, V_{21}, \cdots, V_{2r_2}, \cdots, V_{pr_p})$, then $\mathcal{N}$ is of the form $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$. Let $\mathcal{M} = \mathcal{N}^{-1}$, then $\mathcal{M}$ is of the form $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$, too.

Moreover we put

$$H_2=\left(0\,\middle|\,\begin{array}{c} 0 \\ a_0(x\,;D')-a_0(x\,;\xi'_0/|\xi'_0|)\Lambda_0, \cdots, a_{s-m-1}(x\,;D')-a_{s-m-1}(x\,;\xi'_0/|\xi'_0|)\Lambda_0 \end{array}\right),$$

$$H_3=\left(\begin{array}{c} 0 \\ ib_1(x\,;D'), \cdots, ib_s(x\,;D') \end{array}\right), \qquad C_1=\mathcal{N}_2 H_2 \mathcal{M}=\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right),$$

$$C_2=\mathcal{N}_2 H_2(\mathcal{M}\circ\mathcal{N}-\mathcal{M}\mathcal{N})=\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right), \qquad C_3=\mathcal{N}_2 H_3=\left(\begin{array}{c} 0 \\ \hline * \end{array}\right),$$

and

$$\mathcal{D}=\left(\begin{array}{ccc|cccc} \mu_1 & & & & & & \\ & \ddots & & & & & \\ & & \mu_m & & & & \\ \hline & & & \boxed{\begin{array}{ccc} \lambda_1 & \Lambda_0 & \\ & \ddots & \Lambda_0 \\ & & \lambda_1 \end{array}} & & & \\ & & & & \boxed{\begin{array}{ccc} \lambda_2 & \Lambda_0 & \\ & \ddots & \Lambda_0 \\ & & \lambda_2 \end{array}} & & \\ & & & & & \ddots & \\ & & & & & & \boxed{\begin{array}{ccc} \lambda_p & \Lambda_0 & \\ & \ddots & \Lambda_0 \\ & & \lambda_p \end{array}} \end{array}\right),$$

where $\mathcal{N}_2={}^t(0,\cdots,0,V_{11},\cdots,V_{pr_p})$. Let us remark that C_1, C_2 and C_3 are of order $1, 0$ and 0, respectively, and C_2 and C_3 have sufficiently small operator norms. Then,

$$\mathcal{N}L_0U=D_1\mathcal{N}U-\mathcal{D}\mathcal{N}U-C_1\mathcal{N}U-C_2U-C_3U-\mathcal{N}'_{x_1}U-(\mathcal{N}H-\mathcal{D}\mathcal{N})U$$
$$-\mathcal{N}BU-\mathcal{N}GU-\sum_{j=1}^{n}\mathcal{N}K_jU_j-\mathcal{N}U',$$

and $\|\mathcal{N}L_0U\|\leqslant c\|Lu\|_{-l}$.

4. The estimate of $\mathcal{N}L_0U$. Let us put $\mathcal{N}U=V=(v_j)$. For $(D_1-\mu_j(x;D'))v_j$ $(1\leqslant j\leqslant m)$ by the quasi-local property of pseudo-differential operators, using the weight function $\varphi=(x_1+2h)^{-1}$ we have the following estimates in the same way as in [4] and [5]:

$$\|(D_1-\mu_j(x;D'))v_j\|\geqslant c\|(\Lambda_0+1)^{1-1/m}v_j\|-C(h)\|u\|_{s-l-3},\qquad (1\leq j\leq m).$$

And for $(D_1-\lambda_{j'}(D'))v_j$, modifying the symbol ξ_1 out of supp $[\alpha]$ we have

$$\|(D_1-\lambda_{j'}(D'))v_j\|\geqslant c\|\Lambda v_j\|-C(h)\|u\|_{s-l-3},\qquad (m+1\leqslant j\leqslant s).$$

Here, let $V^{(1)}={}^t(v_1,\cdots,v_m,0,\cdots,0)$, $V^{(2)}={}^t(0,\cdots 0,v_{m+1},\cdots,v_s)$, $U^{(2)}={}^t(0,\cdots,0,u_{m+1},\cdots,u_s)$, and $U_j^{(3)}={}^t(0,\cdots,0,u_{jm},\cdots,u_{js})$ $(0\leqslant j\leqslant n$ and $U_0=U$, $u_{0k}=u_k$, $m\leqslant k\leqslant s)$. Now we have the following

$$\|\mathcal{N}L_0U\|\geqslant c\|(\Lambda_0+1)^{1-1/m}V^{(1)}\|+c\|\Lambda V^{(2)}\|-\delta\|\Lambda V^{(2)}\|$$
$$-c_1\|U^{(2)}\|-c_2\|(\Lambda_0+1)^{-1}U\|-\sum_{j=1}^{n}\|U_j^{(3)}\|-\|U'\|-C(h)\|u\|_{s-l-3}$$
$$\geqslant c\|\alpha u\|_{s-l-1}+c\|(\Lambda_0+1)^{1-1/m}U^{(3)}\|-\sum_{j=1}^{n}\|U_j^{(3)}\|$$
$$-C'\|u\|_{s-l-2}-C(h)\|u\|_{s-l-3}$$

(for sufficiently small h).

In particular if we take $\alpha_i(\xi)$ as a partition of the unity of S_ξ^{n-1} and summing up for i, by the relation $\sum_i\sum_{j=1}^n\|U_j^{(3)}\|\leqslant c\sum_i\|U^{(3)}\|$ we arrive at the following inequality,

$$\|Lu\|_{-l}\geqslant c\|u\|_{-l+s-2}-C(h)\|u\|_{-l+s-3}.$$

Fixing h, the above inequality implies the following

$$\|Lu\|_{-l}\geqslant c\|u\|_{-l+s-2}\qquad \text{for all } u(x)\in\mathcal{D}(B_d)\ (0<d\ll h),$$

(see L. Nirenberg-F. Treves [3], Part II). Q.E.D.

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