31. *On Some Noncoercive Boundary Value Problems for the Laplacian*

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1. Introduction. Let Ω be a bounded domain in $\boldsymbol{R}^n$ with boundary Γ of class C^∞. $\bar{\Omega}=\Omega\cup\Gamma$ is a C^∞-manifold with boundary. Let a, b and c be real valued C^∞-functions on Γ, let $\boldsymbol{n}$ be the unit exterior normal to Γ and let α and β be real C^∞-vector fields on Γ.

We shall consider the following boundary value problem: For given functions f defined on Ω and ϕ defined on Γ find u in Ω such that

$$(*)\qquad \begin{cases} (\lambda-\Delta)u=f & \text{in } \Omega, \\ \mathcal{B}u\equiv a\dfrac{\partial u}{\partial \boldsymbol{n}}+(\alpha+i\beta)u+(b+ic)u=\phi & \text{on } \Gamma. \end{cases}$$

Here $\lambda\geqq 0$ and $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\cdots+\partial^2/\partial x_n^2$. The problem $(*)$ in the case that $\beta(x)\equiv 0$ on Γ, i.e., the *oblique* derivative problem was investigated by many authors (cf. [2], [6], [7], [8]), but the problem $(*)$ in the case that $\beta(x)\not\equiv 0$ on Γ was treated by a few authors, e.g., Vaǐnberg and Grušin [12] (see also [5]), whose results we shall first describe briefly. For each real s, we shall denote by $H^s(\Omega)$ (resp. $H^s(\Gamma)$) the Sobolev space on Ω (resp. Γ) of order s and by $\|\ \|_s$ (resp. $|\ |_s$) its norm.

If $a(x)>|\beta(x)|$ on Γ where $|\beta(x)|$ is the length of the tangent vector $\beta(x)$, then the problem $(*)$ is *coercive* and the following results are valid for all $s>3/2$ (cf. [9]):

i) For every solution $u\in H^t(\Omega)$ of $(*)$ with $f\in H^{s-2}(\Omega)$ and $\phi\in H^{s-3/2}(\Gamma)$ we have $u\in H^s(\Omega)$ and an *a priori* estimate:

$$(1)\qquad \|u\|_s\leqq C_1(\|f\|_{s-2}+|\phi|_{s-3/2}+\|u\|_t)$$

where $t<s$ and $C_1>0$ is a constant depending only on λ, s and t.

ii) If $f\in H^{s-2}(\Omega)$, $\phi\in H^{s-3/2}(\Gamma)$ and (f,ϕ) is orthogonal to some finite dimensional subspace of $C^\infty(\bar{\Omega})\oplus C^\infty(\Gamma)$, then there is a solution $u\in H^s(\Omega)$ of $(*)$.

iii) If $\lambda>0$ is sufficiently large, then we can omit $\|u\|_t$ in the right hand side of (1) and for every $f\in H^{s-2}(\Omega)$ and every $\phi\in H^{s-3/2}(\Gamma)$ there is a unique solution $u\in H^s(\Omega)$ of $(*)$.

If $a(x)\geqq|\beta(x)|$ on Γ and $a(x)=|\beta(x)|$ holds at some points of Γ, then the problem $(*)$ is *noncoercive*. Vaǐnberg and Grušin [12] treated the problem $(*)$ in the case that $n=2$, $a(x)\equiv 1$, $\alpha(x)\equiv 0$, $|\beta(x)|\equiv 1$ on Γ. Under the assumption that $b(x)+ic(x)\neq 0$ on Γ, they proved smoothness, an *a priori* estimate and existence theorems for the solutions of

(∗), which involve a loss of 1 derivative compared with the results i) and ii) (see [12], Theorem 19).

In this note we shall treat the problem (∗) in the case that n is arbitrary and that $a(x) \geqq |\beta(x)|$ on Γ. Under the assumptions expressed in terms of differential geometry such as the second fundamental form of the hypersurface $\Gamma \subset \boldsymbol{R}^n$, the mean curvature of Γ, the divergence of the vector field α and so on (see $(\text{B–1})_s$, $(\text{B–2})_s$, (B–1), (B–2) and (C)), we shall give smoothness, an *a priori* estimate and existence theorems for the solutions of (∗), which involve a loss of 1 derivative compared with the results i), ii) and iii) (Theorem 1 and Theorem 2). Even in the case that $\beta(x) \equiv 0$ on Γ and hence that $a(x) \geqq 0$ on Γ, these results are new (cf. [2], [7], [8]). The details will be given somewhere else.

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2. Preliminaries. Since $\lambda \geqq 0$, for every $\phi \in C^\infty(\Gamma)$ we can uniquely solve the Dirichlet problem:

$$\begin{cases} (\lambda - \Delta) w = 0 & \text{in } \Omega, \\ w = \phi & \text{on } \Gamma, \end{cases}$$

hence we can define the Poisson operator $\mathscr{P}(\lambda)$ by $w = \mathscr{P}(\lambda)\phi$. The mapping $T(\lambda): \phi \to \mathscr{B}\mathscr{P}(\lambda)\phi|_\Gamma$ is a first order pseudodifferential operator on Γ (cf. [5], [6], [12]) and the problem (∗) can be reduced to the study of $T(\lambda)$ by the same argument as the proof of Theorem 2.2 of Taira [11] (cf. [6], [7], [12]). The *principal* symbol of $T(\lambda)$ is

$$(a(x)\,|\xi| - \beta(x,\xi)) + i\alpha(x,\xi)$$

(see [5], § 3). Here $x = (x_1, x_2, \cdots, x_{n-1})$ are some local coordinates in Γ and $\xi = (\xi_1, \xi_2, \cdots, \xi_{n-1})$ are the corresponding dual coordinates in the cotangent space $T^*\Gamma$ and $|\xi|$ is the length of ξ with respect to the Riemannian metric of Γ induced by the natural metric of $\boldsymbol{R}^n$, and $\alpha(x,\xi)$ (resp. $\beta(x,\xi)$) is the principal symbol of the vector field $\alpha(x)/i$ (resp. $\beta(x)/i$).

Let $\Lambda = (1 - \Delta')^{1/2}$ where Δ' is the Laplace-Beltrami operator corresponding to the Riemannian metric of Γ. To apply Theorem 3.1 of Melin [10] to $\operatorname{Re}(\Lambda^{2s-3}T(\lambda))$ where $s \geqq 3/2$ (see Proposition), we have to make a digression. Let $p_1(x,\xi) = a(x)\,|\xi| - \beta(x,\xi)$. Then $p_1(x,\xi) \geqq 0$ on the space of non zero cotangent vectors $T^*\Gamma \backslash 0$ if and only if $a(x) \geqq |\beta(x)|$ on Γ. Hence we assume that $p_1 \geqq 0$ on $T^*\Gamma \backslash 0$. Let $\Sigma = \{\rho \in T^*\Gamma \backslash 0;\ p_1(\rho) = 0\}$. For every tangent vector u of $T^*\Gamma$ at $\rho \in \Sigma$, let v be some vector field on $T^*\Gamma$ equal to u at ρ and define a quadratic form $a_\rho(u,u)$ by the equation:

$$a_\rho(u,u) = (v^2 p_1)_\rho.$$

Since $p_1 \geqq 0$ on $T^*\Gamma \backslash 0$, it follows that $a_\rho(u,u)$ is independent of the choice of v. Let $\tilde{T}_\rho(T^*\Gamma)$ be the complexification of the tangent space $T_\rho(T^*\Gamma)$ of $T^*\Gamma$ at $\rho \in \Sigma$. We consider the symplectic form

$$\sigma=\sum_{1}^{n-1} d\xi_j \wedge dx_j \qquad \text{on } T^*\Gamma$$

and the quadratic form a_ρ as bilinear forms on $\tilde{T}_\rho(T^*\Gamma)\times\tilde{T}_\rho(T^*\Gamma)$. Since σ is non-degenerate, we can define for every $\rho\in\Sigma$ a linear map $A_\rho : \tilde{T}_\rho(T^*\Gamma)\to\tilde{T}_\rho(T^*\Gamma)$ by the equation:

$$\sigma(u, A_\rho v)=a_\rho(u, v), \qquad u, v\in\tilde{T}_\rho(T^*\Gamma).$$

It is easily seen that the spectrum of A_ρ is situated on the imaginary axis, symmetrically around the origin (see [10], § 2). For every $\rho\in\Sigma$, we shall denote by $\widetilde{\mathrm{Tr}}\, H_{p_1}(\rho)$ the sum of the positive elements in $i\cdot$Spectrum (A_ρ) where each eigenvalue is counted with its multiplicity.

The *subprincipal* symbol of $\mathrm{Re}\,(T(\lambda))$ is

$$b(x)-\frac{1}{2}\operatorname{div}\alpha(x)+\frac{1}{2}a(x)(|\xi|^{-2}\,\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x))$$

(cf. [5], § 3). Here div α is the divergence of the vector field α and $M(x)$ is the mean curvature at x of the hypersurface $\Gamma\subset \boldsymbol{R}^n$ and ω_x is the second fundamental form at x of Γ, and $\hat{\xi}\in T_x\Gamma$ is the tangent vector of Γ at x corresponding to $\xi\in T_x^*\Gamma$ by the duality between $T_x\Gamma$ and $T_x^*\Gamma$ with respect to the Riemannian metric of Γ, where $T_x\Gamma$ (resp. $T_x^*\Gamma$) is the tangent (resp. cotangent) space of Γ at x. Further, the subprincipal symbol of $\mathrm{Re}\,(\Lambda^{2s-3}T(\lambda))$ on $\Sigma=\{(x,\xi)\in T^*\Gamma\backslash 0\,;\, a(x)\,|\xi|-\beta(x,\xi)=0\}$ is

$$\left(b(x)-\frac{1}{2}\operatorname{div}\alpha(x)\right)|\xi|^{2s-3}+\frac{1}{2}a(x)(|\xi|^{-2}\,\omega_x(\hat{\xi},\hat{\xi})-(n-1)M(x))\,|\xi|^{2s-3}$$
$$+\frac{1}{2}\{|\xi|^{2s-3}, \alpha(x,\xi)\}-\frac{1}{2}\alpha(x,\xi)\operatorname{div}\delta_\xi(x).$$

Here

$$\{|\xi|^{2s-3}, \alpha(x,\xi)\}=\sum_{j=1}^{n-1}\left(\frac{\partial}{\partial\xi_j}(|\xi|^{2s-3})\frac{\partial}{\partial x_j}\alpha(x,\xi)-\frac{\partial}{\partial\xi_j}\alpha(x,\xi)\frac{\partial}{\partial x_j}(|\xi|^{2s-3})\right)$$

and

$$\delta_\xi(x)=\sum_{j=1}^{n-1}\frac{\partial}{\partial\xi_j}(|\xi|^{2s-3})\frac{\partial}{\partial x_j}$$

is a real C^∞-vector field on Γ defined for $\xi\neq 0$ (cf. [1], Proposition 5.2.1).

3. Results. Applying Theorem 3.1 of Melin [10] to $\mathrm{Re}\,(\Lambda^{2s-3}T(\lambda))$ where $s\geqq 3/2$ and by the same argument as the proof of Theorem 6 of Fujiwara [4], we can obtain

Proposition. *Let $s\geqq 3/2$, $t<s-3/2$. There exist constants $C_3>0$ and C_3' depending only on λ, s and t such that the estimate*

(3) $$\mathrm{Re}\,(\Lambda^{2s-3}T(\lambda)\phi, \phi)\geqq C_3\,|\phi|_{s-3/2}^2-C_3'\,|\phi|_t^2$$

holds for all $\phi\in C^\infty(\Gamma)$ if and only if the following assumptions (A), $(\mathrm{B}\text{–}1)_s$ *and* $(\mathrm{B}\text{–}2)_s$ *hold:*

(A) $$a(x)\geqq|\beta(x)| \qquad \text{on } \Gamma.$$

$(\mathrm{B}\text{–}1)_s$ *At every point $x\in\Gamma$ where $a(x)=0$, the inequality*

$$2b(x)-\operatorname{div}\alpha(x)+\{|\xi|^{2s-3}, \alpha(x,\xi)\}-\alpha(x,\xi)\operatorname{div}\delta_\xi(x)>0$$

holds for all $\xi \in T_x^*\Gamma$ *with* $|\xi|=1$ *(see* (2)).

$(B\text{–}2)_s$ *At every point* $x \in \Gamma$ *where* $a(x)=|\beta(x)|>0$, *the inequality*

$$\widetilde{\mathrm{Tr}}\, H_{p_1}(x,\xi)+2b(x)-\mathrm{div}\,\alpha(x)+a(x)\Big(\omega_x\Big(\frac{\beta(x)}{a(x)},\frac{\beta(x)}{a(x)}\Big)-(n-1)M(x)\Big)$$
$$+\{|\xi|^{2s-3},\alpha(x,\xi)\}-\alpha(x,\xi)\,\mathrm{div}\,\delta_\xi(x)>0$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\beta(x)/a(x) \in T_x\Gamma$ *by the duality between* $T_x^*\Gamma$ *and* $T_x\Gamma$ *with respect to the Riemannian metric of* Γ *(see* (2)).

Furthermore, if $\lambda>0$ *is sufficiently large, then we can omit* $|\phi|_t$ *in the right hand side of* (3).

Remark 1. It follows from the assumption (A) that at every point $x \in \Gamma$ where $a(x)=0$, $\widetilde{\mathrm{Tr}}\, H_{p_1}(x,\xi)=0$ for all $\xi \in T_x^*\Gamma$ with $|\xi|=1$.

Remark 2. If the set $\Gamma_0=\{x \in \Gamma\,;\, a(x)=|\beta(x)|\}$ is an $(n-2)$-dimensional *regular* submanifold of Γ and the vector field α is *transversal* to Γ_0, then for every $s \geqq 3/2$ we can construct a C^∞-function h_s on Γ such that $h_s(x)>0$ on Γ and that the estimate (3) hold with $\Lambda^{2s-3}T(\lambda)$ replaced by $h_s\Lambda^{2s-3}T(\lambda)$ (cf. [8], Lemma 4).

By the same argument as the proof of Theorem 2.2 of Taira [11], we can obtain from Proposition

Theorem 1. *Assume that*

(A) $$a(x) \geqq |\beta(x)| \quad on\ \Gamma$$

and that the assumptions $(B\text{–}1)_s$ *and* $(B\text{–}2)_s$ *hold for some* $s>3/2$.

Then we have:

i)′ *for every solution* $u \in H^{s-1}(\Omega)$ *of* $(*)$ *with* $f \in H^{s-2}(\Omega)$ *and* $\phi \in H^{s-3/2}(\Gamma)$ *we have an a priori estimate:*

(4) $$\|u\|_{s-1} \leqq C_4(\|f\|_{s-2}+|\phi|_{s-3/2}+\|u\|_t)$$

where $t<s-1$ *and* $C_4>0$ *is a constant depending only on* λ, s *and* t;

iii)′ *if* $\lambda>0$ *is sufficiently large, then we can omit* $\|u\|_t$ *in the right hand side of* (4) *and for every* $f \in H^{s-2}(\Omega)$ *and every* $\phi \in H^{s-3/2}(\Gamma)$ *there is a unique solution* $u \in H^{s-1}(\Omega)$ *of* $(*)$.

Remark 3. Further, we can prove that if $f \in H^{s-2}(\Omega)$, $\phi \in H^{s-3/2}(\Gamma)$ and (f,ϕ) is orthogonal to some finite dimensional subspace of $H_0^{-s+2}(\Omega) \oplus H^{-s+3/2}(\Gamma)$ where $H_0^{-s+2}(\Omega)$ is the dual space of $H^{s-2}(\Omega)$, then there is a solution $u \in H^{s-1}(\Omega)$ of $(*)$.

Remark 4. If the assumptions $(B\text{–}1)_s$ and $(B\text{–}2)_s$ hold for *all* $s>3/2$, then by the same argument as the proof of Theorem 7.4 of Egorov and Kondrat'ev [2] we can prove that every solution $u \in H^{s-1}(\Omega)$ of $(*)$ with $f \in H^{s-1}(\Omega)$ and $\phi \in H^{s-1/2}(\Gamma)$ belongs to $H^s(\Omega)$.

Further, applying Theorem 1 of Fediĭ [3] to $T(\lambda)$, we can obtain

Theorem 2. *Assume that*

(A) $$a(x) \geqq |\beta(x)| \quad on\ \Gamma$$

and that the following assumptions (B–1), (B–2) *and* (C) *hold:*

(B–1) *At every point* $x \in \Gamma$ *where* $a(x)=0$, $b(x)>0$.

(B–2) *At every point* $x \in \Gamma$ *where* $a(x)=|\beta(x)|>0$, *the inequality*

$$\widetilde{\mathrm{Tr}}\, H_{p_1}(x,\xi)+2b(x)-\mathrm{div}\,\alpha(x) + a(x)\left(\omega_x\left(\frac{\beta(x)}{a(x)}, \frac{\beta(x)}{a(x)}\right)-(n-1)M(x)\right)>0 \tag{5}$$

holds for $\xi \in T_x^*\Gamma$ *corresponding to* $\beta(x)/a(x) \in T_x\Gamma$.

(C) *There exists a constant* $C_0>0$ *such that the inequality*

$$|d\alpha(x,\xi)|^2 \leqq C_0(a(x)-\beta(x,\xi))$$

holds for all $x \in \Gamma$ *and all* $\xi \in T_x^*\Gamma$ *with* $|\xi|=1$. *Here* $d\alpha$ *is the exterior derivative of* $\alpha(x,\xi)$ *and* $|d\alpha|$ *is the length of the cotangent vector* $d\alpha$ *of* $T^*\Gamma$ *with respect to the natural metric of* $T^*\Gamma$ *induced by the Riemannian metric of* Γ.

Then the assumptions (B–1)$_s$ *and* (B–2)$_s$ *hold for all* s (*hence by Theorem* 1 *we have for all* $s>3/2$ *the results* i)′ *and* iii)′) *and we have for all* $s>3/2$:

i)″ *for every solution* $u \in H^t(\Omega)$ *of* (*) *with* $f \in H^{s-2}(\Omega)$ *and* $\phi \in H^{s-3/2}(\Gamma)$ *where* $t<s-1$, *we have* $u \in H^{s-1}(\Omega)$;

ii)′ *if* $f \in H^{s-2}(\Omega)$, $\phi \in H^{s-3/2}(\Gamma)$ *and* (f,ϕ) *is orthogonal to some finite dimensional subspace of* $C^\infty(\bar{\Omega})\oplus C^\infty(\Gamma)$, *then there is a solution* $u \in H^{s-1}(\Omega)$ *of* (*).

Remark 5. The example of Kato [8] shows that the assumption (C) is necessary for Theorem 2 to be valid.

Remark 6. In the case that $n=2$, the inequality (5) is reduced to the following inequality (6):

$$\widetilde{\mathrm{Tr}}\, H_{p_1}(x,\xi)+2b(x)-\mathrm{div}\,\alpha(x)>0, \tag{6}$$

since

$$\omega_x\left(\frac{\beta(x)}{a(x)}, \frac{\beta(x)}{a(x)}\right)-(n-1)M(x)=0.$$

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