50. *On a Problem of E. L. Stout*

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1. Introduction. The following very interesting theorem of T. Radó [3] was proved by many mathematicians (H. Behnke-K. Stein, H. Cartan, I. Glicksberg, M. Goldstein and T. R. Chow, E. Heinz, R. Kaufman and T. Radó, etc.).

Theorem of Radó. *Let $f(z)$ be a complex-valued continuous function defined in $\{|z|<1\}$. If $f(z)$ is analytic in each component of $\{|z|<1\}-f^{-1}(0)$, then $f(z)$ is analytic in $\{|z|<1\}$.*

On the other hand, E. L. Stout [5] proved the possibility of replacing the set $\{|z|<1\}-f^{-1}(0)$ by $\{|z|<1\}-f^{-1}(E)$ where E is a set of capacity zero. Moreover, he proposed another possibility of $\{|z|<1\}-f^{-1}(0)$ by $\{|z|<1\}-f^{-1}(E)$ where E is a set of positive capacity. In this paper, the present author will give an answer to this problem under some condition.

2. Notation and terminology. Let G be a $n+1$-ply connected region on an open Riemann surface R whose boundary consists of $n+1$ rectifiable closed analytic Jordan curves $C_0, C_1, \cdots, C_n$, where C_0 contains $C_1, \cdots, C_n$ in its interior. Let ω be the harmonic measure in G with boundary values 0 on C_0 and 1 on $C_1, \cdots, C_n$. We call $\mu=2\pi/D_G(\omega)$ the harmonic modulus of G where $D_G(\omega)$ is the Dirichlet integral of ω over G.

Proof of the Theorem. **Lemma** (Sario) (cf. [4]). *Let R be an open Riemann surface. If there exists a normal exhaustion $\{R_n\}$ satisfying $\sum_{n=1}^{\infty}\mu_n^*=\infty$, where μ_n^* is the minimum harmonic modulus of connected components of R_n-R_{n-1}, then R belongs to O_{AD}.*

We shall prove

Theorem. *Let U be an open unit disk $\{|z|<1\}$ and F be a compact set in the complex plane $\boldsymbol{C}$. Let $f(z)$ be a complex-valued continuous function on $\overline{U}$. Set $E=f^{-1}(F)$. Suppose f is analytic in each component of $\overline{U}-E$ and the valence function $n_f(w)$ is finite. If $\hat{C}-F$ belongs to O_{AD} in the sense of Sario ($\hat{C}$ is the one point compactification of $\boldsymbol{C}$), then the set E is of class N_D.*[1] *Moreover if $D_{U-E}(f)<\infty$, then f is analytic in $\overline{U}$ and $D_U(f)<\infty$.*

Proof. First, suppose $n_f(w)$ is bounded and $n_f(w)\leqq N_f$. Let $\{R_n\}$

1) See [1].

be a normal exhaustion of $\hat{C}-F$ and let $R_n-\bar{R}_{n-1}=\bigcup_{i=1}^m R_n^{(j)}$ $(m=m(n))$ where $\{R_n^{(i)}\}$ are connected components of $R_n-\bar{R}_{n-1}$. Let ω_n be the harmonic measure in $R_n-\bar{R}_{n-1}$ with boundary values 0 on ∂R_{n-1} and 1 on ∂R_n. Let $f^{-1}(R_n^{(i)})=\bigcup_{j=1}^l R_n^{(ij)}$ $(l=l(n,i))$ where $\{R_n^{(ij)}\}$ are connected components of $f^{-1}(R_n^{(ij)})$. Then $\omega_n\circ f$ is harmonic in each $R_n^{(ij)}$ and is equal to 0 on $\partial(f^{-1}(R_{n-1}))$ and 1 on $\partial(f^{-1}(R_n))$. Let $\hat{U}$ be the double of U about ∂U. We construct a function $\hat{\omega}_n$ on $\hat{R}_n^{(ij)}=\{\overline{R_n^{(ij)}\cup R_n^{(ij)*}}\}^\circ$ in the following where $R_n^{(ij)*}$ is the symmetric set of $R_n^{(ij)}$ with respect to the origin.

$$\hat{\omega}_n(z)=\begin{cases}(\omega_n\circ f)(z) & z\in\overline{R_n^{(ij)}}\\ (\omega_n\circ f)(z) & z^*\in\overline{R_n^{(ij)*}}.\end{cases}$$

Then $\hat{\omega}_n$ is a Dirichlet function[2] on $\hat{R}_n^{(ij)}$. Let ω_n^* be the harmonic measure in $\hat{R}_n^{(ij)}$ with boundary value $\hat{\omega}_n$ on $\partial\hat{R}_n^{(ij)}$. By the Dirichlet principle, we have

$$\begin{aligned}D_{\hat{R}_n^{(ij)}}(\omega_n^*)&\leqq D_{\hat{R}_n^{(ij)}}(\hat{\omega}_n)\\ &=2D_{R_n^{(ij)}}(\omega_n\circ f)\\ &\leqq 2N_f D_{R_n^{(i)}}(\omega_n).\end{aligned}$$

Then

$$\frac{1}{2N_f}\cdot\frac{2\pi}{D_{R_n^{(i)}}(\omega_n)}\leqq\frac{2\pi}{D_{R_n^{(ij)}}(\omega_n^*)}.$$

Hence we get

$$\frac{1}{2N_f}\cdot\mu_n^{(i)}\leqq\nu_n^{(ij)},$$

where $\mu_n^{(i)}$ is the harmonic modulus of $R_n^{(i)}$ and $\nu_n^{(ij)}$ is the harmonic modulus of of $\hat{R}_n^{(ij)}$. Then it holds

$$\frac{1}{2N_f}\mu_n^*\leqq\nu_n^*,$$

where $\mu_n^*=\min\mu_n^{(i)}$ and $\nu_n^*=\min\nu_n^{(ij)}$, and

$$\infty=\frac{1}{2N_f}\sum_{n=1}^{\infty}\mu_n^*\leqq\sum_{n=1}^{\infty}\nu_n^*.$$

By the Lemma, the set $U-(E\cup E^*)$ belongs to O_{AD} in the sense of Sario where E^* is the symmetric set of E. Hence the set $E\cup E^*$ is of class N_D, which completes the proof.

Secondly, suppose $n_f(w)$ is unbounded. Set $F_n=\{w: n_f(w)\leqq n\}\cap F$ $(n=0,1,2,\cdots)$ and $E_n=f^{-1}(F_n)$. Then E_n belongs to N_D and $E=\bigcup_{n=1}^{\infty}E_n$ belongs to N_D. This completes the proof.

Remark. *In the above proof we used the following theorem: If $\{E_n\}$ is a countable family, with compact union E, of AD-removable sets in a closed Riemann surface R then E is again an AD-removable set* (*cf.* [4]).

2) See [2].

References

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