47. *Local Theory of Fuchsian Systems. I*

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1. Introduction. In this paper, we consider a completely integrable system

$$dX=\Big(\sum_{i=1}^{n}\frac{P_i(x)}{x_i}dx_i\Big)X, \tag{1}$$

where $P_i(x)$, $i=1,\cdots,n$ is an $m\times m$ matrix holomorphic at $x=0$, say

$$P_i(x)=\sum_{k\geq 0}P_{i,k}x^k. \tag{2}$$

Here k denotes a multi-index $(k_1,\cdots,k_n)$, k_i a nonnegative integer, $0=(0,\cdots,0)$, and $x^k=x_1^{k_1},\cdots,x_n^{k_n}$. For two multi-indices k and l, "$k\geq l$" means "$k_i\geq l_i$ for all i" and "$k>l$" means "$k\geq l$ and $k_i>l_i$ for some i". We propose to find out the dominant coefficients in $\{P_{i,k}\}$ which determine the local behavior of the solution of (1).

A change of variables $X=U(x)Y$ with $U(x)$ invertible holomorphic at $x=0$, transforms (1) into the system of the form

$$dY=\Big(\sum_{i=1}^{n}\frac{Q_i(x)}{x_i}dx_i\Big)Y \tag{3}$$

with

$$\sum_{i=1}^{n}\frac{Q_i(x)}{x_i}dx_i=U(x)^{-1}\Big(\sum_{i=1}^{n}\frac{P_i(x)}{x_i}dx_i\Big)U(x)-U(x)^{-1}dU(x). \tag{4}$$

First, we determine $U(x)$ in such a way that (3) has a '*reduced*' form, of which the definition is given in Section 4. Next, we show that by a suitable substitution $Y=x_1^{L_1}\cdots x_n^{L_n}Z$ with $L_i=\mathrm{diag}\,(l_i^1,\cdots,l_i^m)$, where l_i^α is a nonnegative integer, equation (3), which has a '*reduced*' form, can be changed to the equation $dZ=(\sum_{i=1}^n(B_i/x_i)dx_i)Z$ with constant matrices $B_1,\cdots,B_n$.

When preparing this note, we were communicated from T. Kimura, that R. Gérard was solving a problem analogous to ours.

2. Convergence theorem. We prepare a convergence theorem which will be used later.

Theorem 1. *Let*

$$du=\Big(\sum_{i=1}^{n}\frac{F_i(x)}{x_i}dx_i\Big)u \tag{5}$$

be a completely integrable system, where u is a vector and

$$F_i(x)=\sum_{k\geq 0}F_{i,k}x^k,\qquad i=1,2,\cdots,n$$

are matrices convergent and holomorphic for $|x|<\varepsilon$. Then any formal *power series solution of* (5) *converges for* $|x|<\varepsilon$ *and represents a holomorphic solution of* (5).

3. **Integrability condition.** The integrability condition of a system (1) is equivalent to

$$(6)_{i,j:k} \qquad k_j P_{i,k} - k_i P_{j,k} + \sum_{k'+k''=k} [P_{i,k'}, P_{j,k''}] = 0$$

for any $i, j=1, \cdots, n$ and $k=(k_1, \cdots, k_n)$. Here [,] denotes the usual bracket of matrices.

4. **Reduction to a reduced form. Definition.** We say the equation $dX=(\sum_{i=1}^n (P_i(x)/x_i)dx_i)X$ is '*reduced*' with respect to $(k, (\alpha, \beta))$ if $a_i^{\alpha\alpha}-a_i^{\beta\beta}-k_i \neq 0$ for some i implies $p_{j,k}^{\alpha\beta}=0$ for all $j=1, 2, \cdots, n$, where $k=(k_1, \cdots, k_n)$, $P_i(x)=\sum_{k\geq 0} P_{i,k}x^k$, $P_{i,k}=(P_{i,k}^{\alpha\beta})$ and $P_{i,0}=(a_i^{\alpha\beta})$. Furthermore we say the equation has a '*reduced*' form if it is reduced with respect to all $(k, (\alpha, \beta))$.

First we shall determine the coefficient U_k of $U(x)=\sum_{k\geq 0} U_k x^k$ such that the transformed equation has a '*reduced*' form.

4.1. **Formal reduction.** We decompose $U(x)=\sum_{k\geq 0} U_k x^k$ as follows;

$$U(x)=U_0 \cdot U_1(x) \cdots U_N(x) \cdots$$

where U_0 is a nonsingular constant matrix and

$$U_N(x)=U_N^{(1,m)}(x) \cdot U_N^{(2,m)}(x) \cdots U_N^{(m,m)}(x) \cdot U_N^{(1,m-1)}(x) \times U_N^{(2,m-1)}(x) \cdots U_N^{(1,1)}(x) \cdot U_N^{(2,1)}(x) \cdots U_N^{(m,1)}(x)$$

with

$$U_N^{(\alpha,\beta)}=I+\sum_{|k|=N} U_k^{(\alpha,\beta)} x^k.$$

Here $|k|=\sum_{i=1}^n k_i$ and $U_k^{(\alpha,\beta)}$ is a constant matrix, of which the (γ, δ) component is zero except for $(\gamma, \delta)=(\alpha, \beta)$: the (α, β) component will be denoted by $u_k^{\alpha\beta}$. We determine $U_0, U_1^{(1,m)}(x), U_1^{(2,m)}(x), \cdots U_N^{(\alpha,\beta)}(x), \cdots$ successively.

Since $P_{i,0}$ of $P_i(x)$ in (2) is mutually commutative by the integrability condition $(6)_{i,j:0}$, we can choose a nonsingular matrix U_0 such that

$$(7) \quad \begin{cases} \text{(i)} \quad A_i=U_0^{-1}P_{i,0}U_0,\ i=1, \cdots, n \text{ is lower triangular,} \\ \text{(ii)} \quad \text{if } a_i^{\alpha\alpha}-a_i^{\beta\beta} \neq 0 \text{ for some } i, \text{ then } a_j^{\alpha\beta}=0 \text{ for all } j=1, \cdots, n, \end{cases}$$

where $A_i=(a_i^{\alpha\beta})$. We note that the transformed equation by U_0 is '*reduced*' with respect to $(0, (\alpha, \beta))$ for all α, β. Furthermore, using the notation $(\gamma, \delta)<(\alpha, \beta)$ for $\delta>\beta$ or $\delta=\beta$, $\gamma<\alpha$, we have

Proposition 1 (Induction process). *Assume that the completely integral system* $dX=(\sum_{i=1}^n (P_i^{(x)}/x_i)dx_i)X$, $P_i(x)=\sum_{k\geq 0} P_{i,k}x^k$, *is* '*reduced*' *with respect to* $(k, (\gamma, \delta))$ *both for the cases when* $k(|k|<N)$ *and* (γ, δ) *is arbitrary and when* $k(|k|=N)$, $(\gamma, \delta)<(\alpha, \beta)$. *Assume further* $P_{i,0}=A_i=(a_i^{\alpha\beta})$ *satisfies the condition* (7). *Then we can choose a transformation* $=U_N^{(\alpha,\beta)}(x)Y$ *such that a consequent equation* $dY=(\sum_{i=1}^n (Q_i(x)/x_i)dx_i)Y$, $Q_i(x)=\sum_{k\geq 0} Q_{i,k}x^k$, *is* '*reduced*' *with respect to* $(k, (\gamma, \delta))$ *both for the* *cases when* $k(|k|<N)$ *and* (γ,δ) *is arbitrary and when* $k(|K|=N)$, (γ,δ) $\leq(\alpha,\beta)$. *Furthermore*

$$Q_{i,k}=P_{i,k} \quad \text{for } k(|k|<N)\ i=1,2,\cdots,n, \tag{8}$$

$$q_{i,k}^{\gamma\delta}=P_{i,k}^{\gamma\delta} \quad \text{for } k(|k|=N),\ (\gamma,\delta)<(\alpha,\beta). \tag{9}$$

If $a_{i_0}^{\alpha\alpha}-a_{i_0}^{\beta\beta}-k_{i_0}\neq 0$ *for some* i_0, *then the value of* $u_k^{\alpha\beta}(|k|=N)$ *is determined by*

$$(a_{i_0}^{\alpha\alpha}-a_{i_0}^{\beta\beta}-k_{i_0})u_k^{\alpha\beta}+P_{i_0,k}^{\alpha\beta}=0. \tag{10}$$

Apply the above proposition to determine $U_N^{(\alpha,\beta)}(x)$, by regarding the equation for X in the proposition as the equation transformed from (1) by $U_0\cdot U_1^{(1,m)}\cdots U_N^{(\alpha-1,\beta)}(x)$. Then

Theorem 2. *There exists a formal power series* $\sum_{k\geq 0}U_kx^k$ *with* $\det U_0\neq 0$, *such that the formal substitution* $X=(\sum_{k\geq 0}U_kx^k)Y$ *changes the system* (1) *into the system which has a 'reduced' form.*

Remark 1. Although $\sum_{k\geq 0}U_kx^k$ in Theorem 2 is not uniquely determined, it contains only finite number of undetermined parameters.

Remark 2. In Theorem 2, if no two eigenvalues of $P_{i,0}$ differ by an integer for each i, we can choose $\sum_{k\geq 0}U_kx^k$ with $U_0=I$ such that, by $X=(\sum_{k\geq 0}U_kx^k)Y$, (1) is changed to the equation of the form $dY=(\sum_{i=1}^n(P_{i,0}/x_i)dx_i)Y$. In this case, $U_k(k>0)$ is uniquely determined.

4.2. Analytic reduction. By Theorem 1, we can prove the convergence of $U(x)=\sum_{k\geq 0}U_kx^k$ in Theorem 2. Thus

Theorem 3. *Given any completely integrable system* (1), *we have a convergent series* $U(x)=\sum_{k\geq 0}U_kx^k$ *with* $\det U_0\neq 0$ *such that the transformation* $X=U(x)Y$ *takes* (1) *into*

$$dy=\sum_{i=1}^n\left(\frac{Q_i(x)}{x_i}dx_i\right)Y, \tag{11}$$

where

(i) $Q_i(x)=A_i+\sum_{k>0}Q_{i,k}x^k$ *(finite sum)*, A_i *being lower triangular,*

(ii) *the* (α,β) *component* $q_i^{\alpha\beta}(x)$ *of* $Q_i(x)$ *is a monomial of* x,

$$q_i^{\alpha\beta}(x)=q_i^{\alpha\beta}\cdot x_1^{k_1}\cdots x_n^{k_n},$$

with $k_\mu=a_\mu^{\alpha\alpha}-a_\mu^{\beta\beta}$, $\mu=1,\cdots,n$. $q_i^{\alpha\beta}(x)$ *can be nonzero only if* $a_\mu^{\alpha\alpha}-a_\mu^{\beta\beta}$ *is a nonnegative integer* k_μ *for all* $\mu=1,2,\cdots,n$.

5. Singular transformation. Consider the '*reduced*' equation (11) in Theorem 3. Let L_i be a diagonal matrix (l_i^α). A singular transformation $Y=x_1^{L_1}\cdots x_n^{L_n}Z$ changes (11) into $dZ=(\sum_{i=1}^n(B_i(x)/x_i)dx_i)Z$, where $b_i^{\alpha\beta}(x)=q_i^{\alpha\beta}(x)x_1^{l_1^\beta-l_1^\alpha}\cdots x_n^{l_n^\beta-l_n^\alpha}-\delta_\beta^\alpha l_i^\alpha$. Here, $B_i(x)=(b_i^{\alpha\beta}(x))$, $Q_i(x)=(q_i^{\alpha\beta}(x))$ and δ_β^α denotes the Kronecker symbol.

We shall show that $b_i^{\alpha\beta}(x)$ becomes constant by choosing nonnegative integers l_i^α suitably. We classify $\{a_i^{\alpha\alpha}\}_{\alpha=1,\dots,m}$ so that $a_i^{\alpha\alpha}$ and $a_i^{\beta\beta}$ belong to the same class iff $a_i^{\alpha\alpha}-a_i^{\beta\beta}$ is an integer. We denote by $[a_i^{\alpha\alpha}]$ the class of $a_i^{\alpha\alpha}$. For every $a_i^{\alpha\alpha}$, we define $a_i^{\alpha_0\alpha_0}$ as a member of $[a_i^{\alpha\alpha}]$ which has the minimum real part. Then by taking $l_i^\alpha=\alpha_i^{\alpha\alpha}-a_i^{\alpha_0\alpha_0}$, $b_i^{\alpha\beta}(x)$ becomes a constant $b_i^{\alpha\beta}$ by virtue of the properties (i), (ii) in Theorem 3. Thus we have

Theorem 4. *By a change of variables* $Y=x_1^{L_1}\cdots x_n^{L_n}Z$ *with* $L_i=\mathrm{diag}(l_i^1,\cdots l_i^m)$, l_i^α *nonnegative integer, the 'reduced' equation in Theorem* 3 *can be transformed to*

$$dZ=\left(\sum_{i=1}^{n}\frac{B_i}{x_i}dx_i\right)Z,$$

where B_i *is a constant matrix given by*

$$B_i=A_i-L_i+\sum_{k>0}Q_{i,k} \qquad (\textit{finite sum})$$

and satisfies $[B_i,B_j]=0$, $i,j=1,\cdots,n$.

Remark 3. L_i in Theorem 4 is not uniquely determined, but it is unique up to integers in the following sense: Let L_i and L_i' be two diagonal matrices stated in Theorem 4, then $l_i^\alpha-l_i'^\alpha=l_i^\beta-l_i'^\beta$ for any α, β with $[a_i^{\alpha\alpha}]=[a_i^{\beta\beta}]$.

6. Main Theorems. Combining Theorem 3 and Theorem 4, we have

Theorem 5. *Given any completely integrable system* (1) *where* $P_i(x)$ *is holomorphic at* $x=0$, *we have a nonsingular matrix* $U(x)$ *holomorphic at* $x=0$ *and a diagonal matrix* L_i, $i=1,\cdots,n$ *of which the components are nonnegative integers such that the transformation* $X=U(x)x_1^{L_1}\cdots x_n^{L_n}Z$ *changes* (1) *into*

$$dZ=\left(\sum_{i=1}^{n}\frac{B_i}{x_i}dx_i\right)Z,$$

where B_i, $i=1,\cdots,n$ *is a constant matrix satisfying* $[B_i,B_j]=0$ *for all* $i,j=1,\cdots,n$. *The matrices* B_i, L_i *and the coefficients of the power series for* $U(x)$ *can be concretely calculated by algebraic operations. And the eigenvalues of* B_i+L_i *coincide with those of* $P_i(0)$.

By the same argument as in the proof of Theorem 5, we can obtain

Theorem 6. *Given a completely integrable system*

$$dX=\left(\sum_{i=1}^{\nu}\frac{P_i(x)}{x_i}dx_i+\sum_{i=\nu+1}^{n}P_i(x)dx_i\right)X, \tag{12}$$

where $P_i(x)$, $i=1,\cdots,n$ *is an* $m\times m$ *matrix holomorphic at* $x=0$, *we have a transformation* $X=U(x)x_1^{L_1}\cdots x_\nu^{L_\nu}Z$ *which changes* (12) *to* $dZ=(\sum_{i=1}^{\nu}(B_i/x_i)dx_i)Z$ *where* $U(x)$, L_i *and* B_i, $i=1,\cdots,\nu$ *satisfy the same condition as in Theorem* 5. *Furthermore,*

$$U(x)=(I+\sum_{\substack{k_{\nu+1}+\cdots+k_n\geq 1\\ k\geq 0}}V_kx^k)(W_0+\sum_{k_1+\cdots+k_\nu\geq 1}W_{k_1,\cdots,k_\nu}x_1^{k_1}\cdots x_\nu^{k_\nu})$$

The details will be published elsewhere.

References

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