89. *A Weak Solution for the Modified Frankl' Problem*

By Megumi SAIGO
Kumamoto University

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In [2] the modified Frankl' problem for equations of mixed type was proposed and the maximum principle for the problem was proved. In this note we shall consider the system to which the Tricomi equation is reduced, construct a priori estimate by applying the ABC method [1], [3], and show the existence of a weak solution for the problem.

From the required conditions for auxiliary functions the problem must be considered on Hilbert spaces with weights which are singular on the parabolic line of the system and at a special point on it. In order to determine degrees of the weights the electronic computer FACOM 230–25 at Kumamoto University is supplementarily used. Then it can be found that such weights are restricted to peculiar ones for the system. Results for the other equations of mixed type will be published elsewhere.

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1. Modified Frankl' problem. Let Ω be a domain in the (x, y)-plane surrounded by curves $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_+$ and Γ_- as follows: Γ_0 is a segment of the x-axis located between $A(1, 0)$ and $D(d, 0)$, where $d>1$. Γ_2 is a curve in $y<0$ issuing from A with the slope $dx/dy=\sqrt{-y}$ (one of the characteristics of (1) below), and let the intersection of this curve and the y-axis be $C(0, -c)$. Γ_1 is a Jordan arc in $y>0$ joining D and $B(0, c)$. Γ_+ and Γ_- are segments OB and OC of the y-axis, respectively.

Let us consider the following problem for the unknown $u=(u_1, u_2)$:

$$Lu=\begin{pmatrix} y & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial u}{\partial x}+\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial u}{\partial y}=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \text{in } \Omega, \tag{1}$$

$$\begin{cases} u_1n_2-u_2n_1=0 \text{ on } \Gamma_1, \ u_1=0 \text{ on } \Gamma_0, \\ u_2(0, y)+u_2(0, -y)=0 & \text{for } 0<y\leq c, \\ u_1(0, y)+l(y)u_2(0, y)=0 & \text{for } 0<y\leq c \text{ and} \\ u_1(0, y)+m(y)u_2(0, y)=0 & \text{for } -c\leq y<0, \end{cases} \tag{2}$$

where the functions $f_1=f_1(x, y)$, $f_2=f_2(x, y)$, $l(y)$ and $m(y)$ are continuous and $n=(n_1, n_2)$ is the outer normal on Γ_1.

Let $r=\sqrt{9(x-1)^2+4|y|^3}$, and let α be a real number. Let H_α be a class of pairs of measurable functions $u=(u_1, u_2)$ with the norm

$$\|u\|_\alpha^2=\iint_\Omega r^\alpha(u_1^2+u_2^2)dxdy<+\infty$$

and the inner product

$$(u,v)_\alpha=\iint_\Omega r^\alpha(u_1v_1+u_2v_2)dxdy.$$

The adjoint problem for the problem (1), (2) with respect to the inner product in H_α is such that for $v=(v_1,v_2)$

$$L^*v=\begin{pmatrix}(L^*v)_1\\(L^*v)_2\end{pmatrix}=\begin{pmatrix}-y & 0\\ 0 & 1\end{pmatrix}\frac{\partial v}{\partial x}+\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\frac{\partial v}{\partial y} -\frac{1}{r^\alpha}\begin{pmatrix}(yr^\alpha)_x & (r^\alpha)_y\\ (r^\alpha)_y & -(r^\alpha)_x\end{pmatrix}v=\begin{pmatrix}g_1\\g_2\end{pmatrix} \quad \text{in } \Omega, \tag{3}$$

$$\begin{cases} v_1=0 \text{ on } \Gamma_0\cup\Gamma_1,\ \sqrt{-y}v_1+v_2=0 \text{ on } \Gamma_2 \quad \text{and}\\ yl(y)v_1(0,y)+ym(-y)v_1(0,-y)+v_2(0,y)-v_2(0,-y)=0\\ \qquad \text{for } 0<y\leq c.\end{cases} \tag{4}$$

2. A priori estimate. Let $a=a(x,y)$ and $b=b(x,y)$ be functions to be determined later, which are continuous in $\bar{\Omega}$ and continuously differentiable in Ω. Consider $I_0=2(L^*v,\Phi v)_\alpha$, where $\Phi=\begin{pmatrix}a & b\\ -yb & a\end{pmatrix}$ and $v=(v_1,v_2)$ satisfies the condition (4). By virtue of the Green theorem I_0 equals

$$\iint_\Omega r^{2\alpha}[Av_1^2+2Cv_1v_2+Bv_2^2]dxdy + \int_{\partial\Omega} r^\alpha[(-yav_1^2-2ybv_1v_2+av_2^2)n_1+(ybv_1^2-2av_1v_2-bv_2^2)n_2]ds, \tag{5}$$

where $A=(ya/r^\alpha)_x-(yb/r^\alpha)_y$, $B=-(a/r^\alpha)_x+(b/r^\alpha)_y$ and $C=(a/r^\alpha)_y+(yb/r^\alpha)_x$. If the functions a,b could be chosen in such a way that the conditions i) $A\kappa_1^2+2C\kappa_1\kappa_2+B\kappa_2^2$ is positive definite on Ω, where $(\kappa_1,\kappa_2)\in R^2$, ii) $b\geq 0$ on Γ_0, iii) $an_1-bn_2\geq 0$ on Γ_1, and iv) $p_1\kappa_1^2+p_2\kappa_2^2+p_3\kappa_3^2+2p_4\kappa_1\kappa_2+2p_5\kappa_1\kappa_3+2p_6\kappa_2\kappa_3\geq 0$ for $x=0$, $0<y\leq c$ and for $(\kappa_1,\kappa_2,\kappa_3)\in R^3$, where $p_1=y(a-y\bar{a}l^2)$, $p_2=-(a+\bar{a})$, $p_3=-y(\bar{a}+2y\bar{b}\bar{m}+y\bar{a}\bar{m}^2)$, $p_4=y(b-\bar{a}l)$, $p_5=-y^2l(\bar{b}+\bar{a}\bar{m})$, $p_6=-y(\bar{b}+\bar{a}\bar{m})$, $a=a(0,y)$, $\bar{a}=a(0,-y)$ etc. might be satisfied, then I_0 would be greater than some integral on Ω, the integrand of which is a sum of v_1^2 and v_2^2 with positive coefficients.

In order to construct such functions a,b, letting

$$3|x-1|=r\cos\varphi=r\eta, \qquad 2|y|^{3/2}=r\sin\varphi=r\xi, \tag{6}$$

we assume the form

$$a=fr^{\lambda+\alpha}\xi^\mu\eta^\nu, \qquad b=gr^{\rho+\alpha}\xi^\sigma\eta^\tau, \tag{7}$$

where $f,g,\lambda,\mu,\nu,\rho,\sigma$ and τ are real parameters. By substituting (7) into the conditions i)~iv) we have the conditions for the parameters. It must be noted that these parameters are distinct according to the domains $\Omega_1=\Omega\cap\{0<x<1,y>0\}$, $\Omega_2=\Omega\cap\{x>1,y>0\}$ and $\Omega_3=\Omega\cap\{y<0\}$, so that we shall use the suffix i for the respective parameters corresponding to the domain Ω_i $(i=1,2,3)$ except λ and ρ, since the order of the singularity at (1, 0) must be equal in each domain. Also set $\rho=\lambda$

$-1/3$ so that the same exponent of r may be bracketed in A, B and C.

Now, in order to determine the parameters such that the functions a, b in (7) satisfy the conditions i) ~ iv) and are continuous at the boundaries of the contiguous domains Ω_1 and Ω_2 or Ω_1 and Ω_3, we tried to seek them by dint of the electronic computer. We choose $1/3$ as the step length for computations for the exponents $\lambda, \mu, \nu, \sigma$ and τ, and divide the intervals of ξ and η in such a way that the step lengths of ξ^2 and η^2 equal $1/4$. After all only one set of the parameters in Ω_2 was found, namely, $f_2=+1$, $\lambda=-1$, $\mu_2=0$, $\nu_2=1$, $\sigma_2=2/3$ and $\tau_2=0$. Also if we take $g_2=-2^{1/3}$, then $C=0$ exactly. Thus we have

$$a=r^{\alpha-1}\eta, \qquad b=-2^{1/3}r^{\alpha-4/3}\xi^{2/3} \tag{8}$$

in Ω_2. Similarly we have in Ω_1

$$a=-r^{\alpha-1}\eta, \qquad b=-2^{1/3}r^{\alpha-4/3}\xi^{2/3}. \tag{9}$$

Thus in Ω_1 and Ω_2

$$A=2^{-2/3}r^{-4/3}\xi^{2/3}, \qquad B=r^{-2}, \qquad C=0. \tag{10}$$

From (8) and (9) the condition iii) is reduced to $d\varphi/ds\leq 0$ on $\Gamma_1\cap\{0<x<1\}$ and $d\varphi/ds\geq 0$ on $\Gamma_1\cap\{x>1\}$. Then if Γ_1 is star-shaped with respect to $A(1,0)$ in the $(x,\tilde{y})$-plane, where $\tilde{y}=y^{3/2}$, the condition iii) is satisfied, and hereafter we shall assume this for the boundary Γ_1.

So as to choose the parameters in Ω_3, the conditions i) and iv) are simultaneously examined by dint of the computer, where $f_3=-1$, $\mu_3=0$ and $\sigma_3=2/3$ are assigned in advance and the others are moved. Then we obtain different parameters in accordance with various combinations of the functions l and m. For example, if we set $g_3=0$ and $\nu_3=-1$, the condition i) is fulfilled, and for $l=\varepsilon/\sqrt{y}$ and $m=-\delta/\sqrt{-y}$ the condition iv) is satisfactory when the constants ε and δ have the relation $\varepsilon\geq\sqrt{2+\delta^2}$ or $\varepsilon\leq-1-\sqrt{3+\delta^2}$. Note that when $\delta=1$ the fifth in (2) is the modified Frankl' condition in [2] and when $\varepsilon=\delta=0$ the fourth and fifth in (2) are the original Frankl' condition, but this latter case must be omitted, for $l=0$ violates the condition iv).

Thus there holds

$$I_0\geq C_1\iint_\Omega \{r^{2\alpha-4/3}\xi^{2/3}v_1^2+r^{2\alpha-2}v_2^2\}dxdy. \tag{11}$$

On the other hand

$$\begin{aligned} I_0^2\leq C_2\Big[\iint_\Omega \{r^{2\alpha-2/3}\xi^{-2/3}(L^*v)_1^2+r^{2\alpha}(L^*v)_2^2\}dxdy\Big] \\ \times\Big[\iint_\Omega \{r^{2\alpha-4/3}\xi^{2/3}v_1^2+r^{2\alpha-2}v_2^2\}dxdy\Big] \end{aligned} \tag{12}$$

and hence

$$\begin{aligned} &\iint_\Omega \{r^{2\alpha-4/3}\xi^{2/3}v_1^2+r^{2\alpha-2}v_2^2\}dxdy \\ &\qquad\leq C_3\iint_\Omega \{r^{2\alpha-2/3}\xi^{-2/3}(L^*v)_1^2+r^{2\alpha}(L^*v)_2^2\}dxdy. \end{aligned} \tag{13}$$

Now, we have the relation

$$(14)\qquad |(u,v)_1|^2\le C_4\left[\iint_\Omega \{r^{4/3}\xi^{-2/3}u_1^2+r^2u_2^2\}dxdy\right]\cdot\left[\iint_\Omega \{r^{2/3}\xi^{2/3}v_1^2+v_2^2\}\,dxdy\right],$$

which is a Schwarz inequality for the inner product $(u,v)_\alpha$ and the integral forms in (13) and which holds only when $\alpha=1$. Accordingly we shall define the following Hilbert spaces: H_+ and H_- are classes of pairs of measurable functions $u=(u_1,u_2)$ for which the norms $\|u\|_+=\sqrt{(u,u)_+}$ and $\|u\|_-=\sqrt{(u,u)_-}$ are finite, respectively, where

$$(u,v)_+=\iint_\Omega (r^{2/3}\xi^{2/3}u_1v_1+u_2v_2)dxdy \quad\text{or}\quad =\iint_\Omega (|y|\,u_1v_1+u_2v_2)dxdy$$

and

$$(u,v)_-=\iint_\Omega (r^{4/3}\xi^{-2/3}u_1v_1+r^2u_2v_2)dxdy \quad\text{or}\quad =\iint_\Omega r^2\left(\frac{1}{|y|}u_1v_1+u_2v_2\right)dxdy.$$

Denote $\mathring{H}_+$ a subclass of H_+ whose elements v satisfy the condition (4) and $L^*v\in H_-$. Then from (13) and (14) we obtain

Theorem 1. *There hold the inequalities*

$$(15)\qquad \|L^*v\|_-\ge C_5\|v\|_+ \qquad \textit{for } v\in\mathring{H}_+,$$

$$(16)\qquad |(u,v)_1|\le C_6\|u\|_+\|v\|_- \qquad \textit{for } u\in H_+,\ v\in H_-.$$

3. 1-weak solution. Since (15) and (16) are obtained by setting $\alpha=1$, we shall define a weak solution as follows:

Definition. $u\in H_+$ is called a 1-weak solution of the problem (1), (2), if

$$(17)\qquad (u,L^*v)_1=(f,v)_1$$

is valid for all $v\in\mathring{H}_+$, where f is a given function from H_-.

Theorem 2. *For any function $f\in H_-$ there exists a 1-weak solution of the problem* (1), (2).

Proof. By the aid of (15) and (16) we have

$$|(f,v)_1|\le C_5^{-1}C_6\|f\|_-\|L^*v\|_-$$

for every $v\in\mathring{H}_+$. Then $(f,v)_1$ is a bounded linear functional on $L^*(\mathring{H}_+)\subset H_-$. From the Riesz theorem there exists $w\in H_-$ such that $(f,v)_1=(w,L^*v)_-$ for all $v\in\mathring{H}_+$. If we set $u_1=r^{1/3}\xi^{-2/3}w_1$, $u_2=rw_2$ then $u=(u_1,u_2)$ is the 1-weak solution, since $(w,L^*v)_-=(u,L^*v)_1$ and $\|w\|_-=\|u\|_+$.

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