85. *On the Korteweg-de Vries Equation*

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1. Introduction. Gardner, Greene, Kruskal and Miura [1] discovered that the discrete eigenvalues of one-dimensional Schrödinger operators $L(t)=d^2/dx^2+u(x,t)$ are constants in t while the potential $u(x,t)$ varies according to the Korteweg de Vries (KdV) equation:

(1.1) $$u_t+6uu_x+u_{xxx}=0, \qquad -\infty<x,t<+\infty,$$

where the subscripts x, t denote partial differentiations. From this they gave a method of constructing solutions of the KdV equation by means of the inverse scattering problem for $L(t)$. Lax [2] presented a general principle for a family of selfadjoint operators $L(t)$ to be unitary equivalent. Applying this principle he gave another proof of the invariance of the eigenvalues and derived an infinite family of equations (higher order KdV equations) that leave the eigenvalues of $L(t)$ invariant in time. Menikoff [3] gave another criterion for the invariance of the eigenvalues of $L(t)$, which works in the case when $u(x,t)\to-\infty$ as $|x|\to+\infty$. His basic idea is to associate the eigenvalue problem for $L(t)$ the following boundary value problem of parabolic type:

$$G_s=G_{xx}+u(x,t)G,$$
$$\lim_{s\to0} G(x,y,s;t)=\delta(x-y),$$
$$\lim_{|x|\to+\infty} G(x,y,s;t)=0.$$

In the class of periodic functions the author [4] has given another characterization of an infinite family of higher order KdV equations and presented a constructive method of deriving them.

In this paper, our purpose is to derive an infinite family of the (higher order) KdV equations and their conserved densities by means of an improved version of the Menikoff's method. Here we shall consider the periodic boundary value problems.

2. Invariance of the eigenvalues and derivation of the infinite family of (higher order) KdV equations. Let $u(x,t)$ be infinitely differentiable real functions of x and t in $R^1\times R^1$ and periodic with respect to x with period 1. Consider the eigenvalue problem with t considered as a parameter:

(2.1) $$\begin{cases} L(t)\varphi=\varphi_{xx}+u(x,t)\varphi=-\lambda\varphi, \\ \varphi(x,t)=\varphi(x+1,t), \\ \varphi_x(x,t)=\varphi_x(x+1,t). \end{cases}$$

Then, there exists a complete set of (real) normalized eigenfunctions $\varphi_j(x,t)$ and eigenvalues $\lambda_j(t)$, $j=1,2,\cdots$, counted according to their multiplicity.

Let $G(x,y,s;t)$ be the fundamental solution of the problem:

$$(2.2)\qquad \begin{cases} G_s=L(t)G=G_{xx}+u(x,t)G, \\ \lim_{s\to 0} G(x,y,s;t)=\delta(x-y), \\ G(x,y,s;t)=G(x+1,y,s;t), \\ G_x(x,y,s;t)=G_x(x+1,y,s;t). \end{cases}$$

Then, we have

$$(2.3)\qquad G(x,y,s;t)=\sum_{j=1}^{\infty} e^{-\lambda_j(t)s}\varphi_j(x,t)\varphi_j(y,t).$$

Theorem 1 (Menikoff). *The eigenvalues of* (2.1) *are constants as t varies if and only if the function $u(x,t)$ satisfies*

$$(2.4)\qquad \int_0^1 u_t(x,t)G(x,x,s;t)dx=0 \qquad \text{for all } s>0 \text{ and } t.$$

Remark 1. The assertion of Theorem 1 is always valid when the Schrödinger operator $L(t)$ has a complete system of eigenfunctions and eigenvalues.

Theorem 2. *As $s\searrow 0$, we have the following asymptotic expansion:*

$$(2.5)\qquad G(x,x,s;t)\sim\sum_{i=0}^{\infty} s^{-1/2+i}P_i(x,t),$$

where $P_i(x,t)$ is uniquely determined and can be computed explicitly in terms of u and its x-derivatives of order $\leqq 2(i-1)$ in the following way:

$$(2.6)\qquad P_i(x,t)=\frac{1}{2\sqrt{\pi}}\sum_{i\geqq k\geqq 0}\frac{(-1)^{i+k}(2k-1)!!}{2^k(i+k)!}P_{i+k,2k}(x,t),$$

where $P_{j,r}$ is a coefficient of the polynomial:

$$(2.7)\qquad M(\xi)=(2\xi d/dx+d^2/dx^2+u(x,t))^j\cdot 1=\sum_{r=0}^{j} P_{j,r}\xi^r$$

and

$$(2k-1)!!=(2k-1)(2k-3),\cdots,3\cdot 1, \qquad (-1)!!=1.$$

In virtue of the periodicity of $G(x,x,s;t)$ with respect to x, we have

$$\int_0^1 dG(x,x,s;t)/dx\cdot G(x,x,s;t)dx=0 \qquad s>0.$$

From this equality and Theorem 2, we obtain

$$(2.8)\qquad \int_0^1 dP_i(x,t)/dx\cdot G(x,x,s;t)dx=0, \qquad i=0,1,2,\cdots \qquad s>0.$$

Hence, we obtain

Theorem 3. *If $u(x,t)$ evolves according to the equation*

$$(2.9)\qquad \frac{\partial u}{\partial t}=\sum_{i=1}^{M} f_i(t)\frac{\partial}{\partial x}P_i(u,u^{(1)},\cdots,u^{2(i-1)}),$$

where $u^{(r)}=\partial^r u/\partial x^r$, *$M$ is an arbitrary finite positive number and* $f_i(t)$ *are arbitrary smooth functions, then the eigenvalues of* (2.1) *are constant as t varies.*

Examples. We have $P_0=1/2\sqrt{\pi}$, $P_1=-u/2\sqrt{\pi}$,

$$P_2=u^2/4\sqrt{\pi}+u_{xx}/12\sqrt{\pi},$$

and so on. Hence, we have the KdV equation when $i=2$:

$$u_t+12\sqrt{\pi}(P_2)_x=u_t+6uu_x+u_{xxx}=0.$$

Now, from (2.3) and (2.5), we get

$$\sum_{j=1}^{\infty} e^{-\lambda_j(t)s}(\varphi_j(x,t))^2 \sim \sum_{i=0}^{\infty} s^{-1/2+i}P_i(x,t) \tag{2.10}$$

from which it follows that

$$\sum_{j=1}^{\infty} e^{-\lambda_j(t)s} \sim \sum_{i=0}^{\infty} s^{-1/2+i}\int_0^1 P_i(x,t)dx. \tag{2.11}$$

If $u(x,t)$ satisfies the equation (2.9), then the left hand side is independent of t. Hence, the quantities

$$\mathscr{P}_i(t)=\int_0^1 P_i(u,u^{(1)},\cdots,u^{2(i-1)})dx, \qquad i=0,1,2,\cdots$$

are invariant integrals of the equation (2.9). Therefore P_i are conserved densities: Thus we have

Theorem 4. *The* $P_i(u,u^{(1)},\cdots,u^{2(i-1)})$ *are conserved densities of* (2.9) *and uniquely determined by the asymptotic expansion* (2.10).

Detailed proofs and further investigations will appear elsewhere.

References

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[2] Lax, P. D.: Integrals of nonlinear equations of evolution and solitary waves. Comm. Pure App. Math., **21**, 467–490 (1968).

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[4] Tsutsumi, M.: Nonlinear evolution equations (to appear).