116. On Extensions of my Previous Paper "On the Korteweg-de Vries Equation"

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1. Introduction. Previously, in [1] we have proved the following result: *Let* $\{\varphi_j(x;t)\}$ *and* $\{\lambda_j(t)\}$, $j=1,2,\cdots$, *be a complete system of normalized eigenfunctions and eigenvalues, respectively, of the Schrödinger eigenvalue problem in* T^1, T^1 *being a torus, with* t *considered as a parameter*:

$$(1.1)\qquad \begin{cases} \dfrac{d^2}{dx^2}\varphi_j(x;t)+u(x,t)\varphi_j(x;t)=-\lambda_j(t)\varphi_j(x;t), \\ \varphi_j(\cdot,t)\in C^2(T^1), \qquad \text{for } {}^\forall t\in(-\infty,\infty), \end{cases}$$

where $u(x,t)$ *is a real function belonging to* $C^\infty(T^1\times R^1)$. *Then we have the asymptotic expansion*:

$$(1.2)\qquad \sum_{j=1}^{\infty} e^{-\lambda_j(t)s}(\varphi_j(x,t))^2 \sim \sum_{i=0}^{\infty} s^{i-1/2}P_i(u,\partial u/\partial u,\cdots,\partial^{2(i-1)}u/\partial x^{2(i-1)})$$

where P_i *are uniquely determined and can be calculated explicitly in terms of the function* u *and its partial derivatives in* x, *of order* $\leqq 2(i-1)$. *If* $u=u(x,t)$ *evolves according to the equation*

$$(1.3)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=\displaystyle\sum_{i=1}^{M} f_i(t)\frac{\partial}{\partial x}P_i(u,\cdots,\partial^{2(i-1)}u/\partial x^{2(i-1)}), \\ u(x,t)\in C^\infty(T^1\times R^1), \end{cases}$$

where M *is an arbitrary fixed positive integer and* $f_i(t)$ *are arbitrary smooth function of* t, *then the eigenvalues* $\lambda_j(t)$ *of the associated eigenvalue problem* (1.1) *are constants in* t *and every* $P_i(\cdot)$ *appeared in* (1.2) *is the conserved density of* (1.3).

In this note, two extensions of the above result are considered. One is to extend it into $n\times n$ matrix form. The other is to extend it into the case of many space variables.

2. $\boldsymbol{n\times n}$ **matrix form.** Let $U(x,t)$ be a $n\times n$ Hermitian matrix function whose elements belong to $C^\infty(T^1\times R^1)$. Below, we denote the set of such matrix functions by $\boldsymbol{C}^\infty(T^1\times R^1)$. Consider the eigenvalue problem for the following matrix differential equation with t considered as a parameter:

$$(2.1)\qquad \begin{cases} \dfrac{d^2}{dx^2}\Phi+U(x,t)\Phi=-\lambda\Phi, \qquad -\infty<x,t<+\infty, \\ \Phi(\cdot\,;t)\in \boldsymbol{C}^2(T^1) \qquad \text{for all } t\in(-\infty,\infty). \end{cases}$$

There exists a complete system of normalized eigen-matrices $\{\Phi_j(x;t)\}$ and eigenvalues $\{\lambda_j(t)\}$, $j=1,2,\cdots$, counted according to their multiplicity. Let $G(x,y,s;t)$ be the Green matrix of the following problem of parabolic type:

$$(2.2)\quad \begin{cases} \dfrac{\partial}{\partial s}G=\dfrac{\partial^2}{\partial x^2}G+U(x,t)G, \\ \lim\limits_{s\searrow 0} G(x,y,s;t)=\delta(x-y)I, \text{ I being the identity matrix,} \\ G(\cdot,y,s;t)\in C^\infty(T^1), \qquad \text{for all } y,t\in(-\infty,\infty) \text{ and all } s>0. \end{cases}$$

We have

$$(2.3)\qquad G(x,y,s;t)=\sum_{j=1}^{\infty} e^{-\lambda_j(t)s}\Phi_j(x,t)\Phi_j^*(y;t),$$

where the asterisk indicates the conjugate transpose.

Theorem 1. *The eigenvalues of* (2.1) *are constants as t varies if and only if the matrix function $U(x,t)$ satisfies*

$$(2.4)\qquad \int_0^1 \text{trace}\left(\frac{\partial}{\partial t}U(x,t)G(x,x,s;t)\right)dx=0,$$

for all $s>0$ and all $t\in(-\infty,\infty)$.

Theorem 2. *As $s\searrow 0$, we have the following asymptotic expansion*:

$$(2.5)\qquad G(x,x,s;t)\sim\sum_{i=0}^{\infty} s^{i-1/2}P_i(x,t),$$

where $P_i(x,t)$ are $n\times n$ matrices whose elements can be computed explicitly in terms of the elements of $U, \partial U/\partial x, \cdots$ and $\partial^{2(i-1)}U/\partial x^{2(i-1)}$.

Theorem 3. *We have*

$$(2.6)\qquad \int_0^1 \text{trace}\left(\frac{\partial}{\partial x}P_i(x,t)\cdot G(x,x,s;t)\right)dx=0$$

for all $t\in(-\infty,\infty)$ and all $s>0$.

Combining Theorem 1 with Theorem 3, we obtain

Theorem 4. *If $u(x,t)$ evolves according to the equation*

$$(2.7)\qquad \begin{cases} \dfrac{\partial}{\partial t}U=\sum\limits_{i=1}^{M} f_i(t)\dfrac{\partial}{\partial x}P_i(U,\cdots,\partial^{2(i-1)}U/\partial x^{2(i-1)}), \\ U(x,t)\in C^\infty(T^1\times R^1), \end{cases}$$

then, all eigenvalues of (2.1) *are constant in t. Furthermore, the quantities*

$$(2.8)\qquad \int_0^1 \text{trace}\, P_i(U,\cdots,\partial^{2(i-1)}U/\partial x^{2(i-1)})dx, \qquad i=0,1,2,\cdots,$$

are invariant integrals of the equation (2.7).

Example. In an analogous way as that in [1], we have

$$(2.9)\qquad \frac{\partial}{\partial t}U+12\sqrt{\pi}\,\frac{\partial}{\partial x}P_2=\frac{\partial}{\partial t}U+3\frac{\partial}{\partial x}(U^2)+\frac{\partial^3}{\partial x^3}U=0,$$

which is a matrix analogue of the Korteweg-de Vries equation. We consider the case when U is a 2×2 real symmetric matrix:

$U=\begin{pmatrix} a & c \\ c & b \end{pmatrix}$, where a, b and c are real functions. Then, the equation (2.9) is reduced to the system:

$$\frac{\partial}{\partial t}a+3\frac{\partial}{\partial x}(a^2+c^2)+\frac{\partial^3}{\partial x^3}a=0, \tag{2.10}$$

$$\frac{\partial}{\partial t}b+3\frac{\partial}{\partial x}(b^2+c^2)+\frac{\partial^3}{\partial x^3}b=0, \tag{2.10'}$$

$$\frac{\partial}{\partial t}c+3\frac{\partial}{\partial x}[(a+b)c]+\frac{\partial^3}{\partial x^3}c=0. \tag{2.10''}$$

If we choose

$$a=b=-u^2 \quad \text{and} \quad c=\frac{\partial}{\partial x}u,$$

where u is a real function, the equations (2.10) and (2.10′) yield

$$u\left(\frac{\partial}{\partial t}u-6u^2\frac{\partial}{\partial x}u+\frac{\partial^3}{\partial x^3}u\right)=0 \tag{2.11}$$

and the equation (2.10″) is

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}u-6u^2\frac{\partial}{\partial x}u+\frac{\partial^3}{\partial x^3}u\right)=0. \tag{2.12}$$

Thus we have

Theorem 5. *If $u(x,t)$ varies according to the modified Korteweg-de Vries equation:*

$$\frac{\partial}{\partial t}u-6u^2\frac{\partial}{\partial x}u+\frac{\partial^3}{\partial x^3}u=0, \tag{2.13}$$

with

$$u(x,t)\in C^\infty(T^1\times R^1), \tag{2.14}$$

then the eigenvalues of the problem:

$$\frac{d^2}{dx^2}\Phi+\begin{pmatrix} -u^2 & \partial u/\partial x \\ \partial u/\partial x & -u^2 \end{pmatrix}\Phi=-\lambda\Phi, \qquad \Phi\in C^2(T^1) \tag{2.15}$$

are constants in t.

3. Many space variable case. Let $u(x,t)$ be an infinitely differentiable real function defined on $T^n\times R^1$, where T^n denotes the n-torus. Let $\{\varphi_j(x;t)\}$ and $\{\lambda_j(t)\}$, $j=1,2,\cdots$ be a complete system of normalized eigenfunctions and eigenvalues, respectively, of the Schrödinger eigenvalue problem in T^n with t considered as a parameter:

$$\begin{cases} \Delta\varphi_j(x;t)+u(x,t)\varphi_j(x;t)=-\lambda_j(t)\varphi_j(x;t), \\ \varphi_j(\cdot,t)\in C^2(T^n) \qquad \text{for } {}^\forall t\in(-\infty,\infty). \end{cases} \tag{3.1}$$

Let $G(x,y,s;t)$ be the Green function of the following problem of parabolic type:

$$\begin{cases} \frac{\partial}{\partial s}G=\Delta G+u(x,t)G, \\ \lim_{s\searrow 0} G(x,y,s;t)=\delta(x-y), \\ G(\cdot,y,s;t)\in C^2(T^n) \qquad \text{for } {}^\forall y\in R^n,\ {}^\forall s>0 \text{ and } {}^\forall t\in(-\infty,\infty). \end{cases} \tag{3.2}$$

Then, we have

$$G(x, y, s; t)=\sum_{j=1}^{\infty} e^{-\lambda_j(t)s}\varphi_j(x, t)\varphi_j(y, t).$$

Theorem 1. *The eigenvalues $\lambda_j(t)$ of* (3.1) *are constants in t if and only if $u(x, t)$ satisfies*

$$\int_0^1 \cdots \int_0^1 \frac{\partial}{\partial t}u(x, t)G(x, x, s, t)dx=0. \tag{3.3}$$

Theorem 2. *As $s \searrow 0$, we have the asymptotic expansion*:

$$G(x, x, s, t) \sim \sum_{i=0}^{\infty} s^{i-n/2}P_i(x, t) \tag{3.4}$$

where P_i can be calculated in terms of u and their partial derivatives with respect to x, of order $\leqq 2(i-1)$.

Theorem 3. *We have*

$$\int_0^1 \cdots \int_0^1 (\boldsymbol{b}(t)\cdot \nabla P_i(x, t))G(x, x, s, t)dx=0, \qquad i=1, 2, \cdots, \tag{3.5}$$

where $\boldsymbol{b}(t)=(b_1(t), \cdots, b_n(t))$ is an arbitrary real vector function and $\nabla=(\partial/\partial x_1, \cdots, \partial/\partial x_n)$.

Theorem 4. *If $u(x, t)$ evolves according to the equation*

$$\frac{\partial}{\partial t}u=\sum_{i=1}^{M} f_i(t)\boldsymbol{b}(t)\cdot \nabla P_i, \qquad u \in C^\infty(T^n \times R^1), \tag{3.6}$$

where M is an arbitrary positive number and $f_i(t)$ are arbitrary smooth functions, then the eigenvalues of (3.1) *are constants as t varies. Furtheremore, every P_i appeared in* (3.4) *is the conserved density of* (3.6).

Example. We obtain

$$\begin{aligned}&\frac{\partial u}{\partial t}+12\sqrt{\pi}\,\boldsymbol{b}(t)\cdot \nabla P_2\\ &\quad=\frac{\partial u}{\partial t}+\sum_{k=1}^{n} b_k(t)\left(6u\frac{\partial u}{\partial x_k}+\Delta\frac{\partial u}{\partial x_k}\right)=0.\end{aligned} \tag{3.7}$$

Theorem 5. *If $u(x, t)$ evolves according to the equation* (3.7) *with $u(x, t) \in C^\infty(T^n \times R^1)$, then all eigenvalues of* (3.1) *are independent of t. Furthermore, all P_i are conserved densities of* (3.7).

Detailed proofs and further investigations will appear elsewhere.

Reference

[1] Tsutsumi, M.: On the Korteweg-de Vries equation. Proc. Japan Acad., **51**, 399–401 (1975).